Equity Forward Return from Derivatives *

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Abstract

This paper develops a theory of forward returns for an equity index. We obtain the forward returns using information from derivatives markets, including index option prices and gammas, VIX-futures, and prices of VIX-options. We document a pro-cyclical term structure of S&P500 forward returns and a robust short-term reversal pattern. Moreover, by designing and implementing a market-timing strategy, we demonstrate that forward equity returns provide real-time trading signals with substantial economic value.

Keywords: Forward return, index option, VIX-derivatives, autocorrelation, reversal

JEL Classification: G1, G12, G13

"The field of finance can be built, or as I will argue be rebuilt, on the basis of 'observable' magnitudes. I still remember the teasing we financial economists, Harry Markowitz, William Sharpe, and I, had to put up with from the physicists and chemists in Stockholm when we concede that the basic unit of our research, the expected rate of return, was not actually observable."

— (Miller, 1999, page 100)

1 Introduction

As argued by Miller (1999), "simply averaging the returns of the last few years, along the lines of the examples in the Markowitz paper (and later book), won't yield reliable estimates of the return *expected* in the future" (page 97). Since the derivatives market provides forward-looking information related to the expected return, previous studies have successfully derived the *expected spot return* from derivatives markets. For instance, Martin (2017); Chabi-Yo and Loudis (2020) for the aggregate market; Martin and Wagner (2019); Kadan and Tang (2020) for individual stocks; and Kremens and Martin (2019) for currencies.

In this paper, we introduce the notion of a *forward equity return* and establish its relationship to certain derivative securities. Specifically, for any positive number n, the forward return $\mathbb{E}_t[R_{t+n\to t+n+1}]$ is the expectation conditional on time t of the future return $R_{t+n\to t+n+1}$ of the underlying asset over a future time interval from $t+n$ to $t+n+1$. We develop a methodology to measure forward returns implied in information from the derivatives market, including prices of index options and VIX-derivatives. Furthermore, we can derive all higher moments of future returns using derivatives market information. Our approach even reveals new information about spot returns from the derivatives not previously studied in the literature, such as the conditional correlation between two spot returns and the joint distribution between two consecutive returns or two spot returns.[1] Importantly, our approach does not require us to impose any distributional

[1] Remarkable exceptions in the literature include Martin (2021), and Chabi-Yo (2019). We will explain these related studies and the distinct features of our paper.

assumptions on the aggregate equity market.

Ultimately, we can construct an entire term structure of forward returns starting from the expected spot return when $n = 0$.[2] Knowing such a term structure of forward returns, we can investigate how a future return $R_{t+n\to t+n+1}$ *dynamically evolves* with market information available at time t. As an example, what is the autocorrelation coefficient between $R_{t+n\to t+n+1}$ and $R_{t+n+1\to t+n+2}$? By answering this question, we can study the momentum or reversal patten of the equity market from a forward-looking derivative perspective.[3] Similarly, we can design dynamic trading strategies based on the conditional view of the aggregate market's future returns from derivatives.

We first express the equity index's forward return in terms of available VIX-derivatives (VIX-futures and VIX-options). Because of its high volume and vast liquidity, incorporating information from the VIX derivatives market is essential in constructing a complete picture of the equity market. In the expression we derive, VIX derivatives play an analogous role for forward returns as index options play for the expected spot return (see Martin, 2017). All quantities in this expression are observable in real-time. Thus, we can compute these forward returns in real-time as well. Since this expression depends on VIX-derivatives, we call it the *VIX-approach* for forward returns.

We present three applications of the VIX-approach. In the first application, we document a *pro-cyclical* term structure of forward returns: upward sloping in good times but downward sloping in bad times. Several studies have documented the average shape of the term structure of the equity risk premium, which is the expected spot return of dividend strips across different maturities. For instance, Binsbergen, Brandt, and Koije (2012) and Binsbergen and Koijen (2017) document that the equity term structure is downward sloping, on average. Moreover, Gormsen (2021) finds that

[2] Our approach to modeling the term structure of equity forward returns, $\mathbb{E}_t[R_{t+n\to t+n+m}], m > n > 0$, instead of the sequence of expected spot returns $\mathbb{E}_t[R_{t\to t+n}], n > 0$, is conceptually similar to forward rate models compared to spot rate models in fixed income (see, for instance, Heath, Jarrow, and Morton, 1992; Duffie and Singleton, 1999). However, there are substantial differences between equity returns and interest rates; and the forward equity return cannot be derived from the equity spot return. Specifically, spot returns are static, similar to the current yield curve in the fixed income market. In contrast, the term structure of forward returns provides a dynamic movement of the equity return, which resembles the movement of the yield curve.

[3] There has been extensive research about the realized autocorrelation using historical equity index data (see, for instance, Lo and MacKinlay, 1988, 1990; Fama and French, 1988; Poterba and Summers, 1988; Moskowitz, Ooi, and Pedersen, 2012). These studies do not use derivatives.

the equity term structure is downward sloping in good times, but upward sloping in bad times, and thus counter-cyclical. We document a new stylized fact about the shape of the term structure of forward equity returns. The key difference between the previous studies and ours is that, in our setting, we recover expected future returns on the aggregate market, whereas these previous studies focus on the spot return of a dividend-strip (Binsbergen, Brandt, and Koije, 2012; Binsbergen and Koijen, 2017), the futures of the dividend-strip (Bansal, Miller, Song, and Yaron, 2021), or a short-maturity asset in excess of long-maturity asset (Gormsen, 2021). Moreover, we use VIX-derivatives, whereas dividend strips, index options, or asset pricing models are used in previous studies.

In the second application, we use the VIX-approach to compute conditional autocorrelation in real-time. Predicting the expected market return with past return observations has been a challenge for researchers and practitioners for decades. Can past returns forecast future returns? Are the returns of a given stock market index autocorrelated? How can we estimate the unconditional autocorrelation coefficients of market returns? Despite extensive research about the realized autocorrelation using historical data (see, for instance, Lo and MacKinlay, 1988, 1990; Fama and French, 1988; Poterba and Summers, 1988; Moskowitz, Ooi, and Pedersen, 2012), the literature still offers no clear guidance as findings have varied depending on the horizon studied and on the sample frequency selected (Campbell, 2017; Baltussen, van Bekkum, and Da, 2019). Moreover, what remains unclear is how to infer the *forward-looking autocorrelation* perceived by investors, as the true autocorrelation may diverge significantly from zero and fluctuate over time (LeRoy, 1973). Empirically, we document *significantly negative* autocorrelation on the S&P 500 index from index options and VIX-derivatives. For instance, the conditional autocorrelation between two consecutive monthly returns, $corr_t\left(R_{t\to t+1mo}, R_{t+1mo\to t+2mo}\right)$, is on average -20.90% with a t-stat of -18.10. On average, the market autocorrelation on the S&P 500 index is around -20% to -40%, suggesting a robust short-term reversal from the perspective of derivatives.

The third application illustrates the economic value of forward returns from derivatives. Specifically, we construct a reversal signal to trade the market. This signal relies on the real-time autocorrelation identified from the derivatives market in the second application. We find that the reversal signal predicts market downturns well, particularly when the market declines significantly in the next month. Furthermore, we show that the corresponding market timing strategy is conservative and delivers higher Sharpe ratios compared to the buy-and-hold benchmark strategy. Moreover, the economic value of this timing strategy can be substantial during prolonged market downturns. For example, investors are willing to pay as much as 11% per annum to switch from the buy-and-hold strategy to the derivative-based market-timing strategy during the 2008/09 global financial crisis, January 2008−June 2009.

The VIX-approach relies on the assumption that $\theta_t \equiv corr_t^Q(R_{t\to t+1}, R^2_{t+1\to t+2}) = 0$, which states that the *risk-neutral conditional correlation coefficient* between $R_{t\to t+1}$ and $R^2_{t+1\to t+2}$ is zero. This risk-neutral zero correlation assumption seems rather technical and restrictive as our objective is to use available market information alone and avoid using any model assumption about the future return. Moreover, there are no available derivatives in the market yet to reveal this risk-neutral correlation coefficient directly. Therefore, we need to (1) justify the VIX-approach empirically for the equity index and (2) introduce econometric methods to estimate this risk-neutral correlation coefficient.

For this purpose, we derive an alternative expression for forward returns using another set of derivatives data - index option prices and their gammas. A disadvantage of this approach is that it cannot be applied in real-time. Nevertheless, by comparing this expression for forward returns to those derived using the VIX-approach, we can estimate past values of θ_t and analyze its time series properties. In the end, we document that the VIX-approach is reasonably good as the sample average of θ_t is fairly close to zero. Moreover, if we use the estimated θ_t, we derive a more robust expression of forward return. In the latter expression of forward return, we use all historical and current index option derivatives (option prices and gammas) and the VIX-derivatives. In this paper, we refer to this more general approach to estimating forward returns as the *VIX-Gamma approach*.

We implement the VIX-Gamma approach in the market-timing strategy and demonstrate that its economic value is even more significant than in the VIX-approach. For instance, we find that investors are willing to pay 31.153% per annum to switch from the buy-and-hold portfolio to the VIX-Gamma-based trading strategy, more than double the amount for the VIX-approach. Moreover, VIX-Gamma-based trading strategy produces a positive average return and a positive Sharpe ratio, even during the 2008/09 global financial crisis period.

Our study is related to Martin (2021), which reduces the conditional expected future return to the no-arbitrage price of a "forward-start option". Since the forward-start option is traded only in over-the-counter markets, he obtains quoted prices from a sophisticated investment bank for a small number of days. In contrast, we present two new expressions of the expected future return that rely on VIX-derivatives and index options—pricing data for both are publicly available. Notably, in our second application, the autocorrelation based on the exchange-traded derivatives is comparable (in magnitude) to that based on the over-the-counter derivatives in Martin (2021). Moreover, we demonstrate novel implications for the equity term structure, investment trading strategy, and risk-neutral density. Another related working paper is by Chabi-Yo (2019), who derives lower and upper bounds, varying from -28% to -3% with a mean value of -14%, on the market autocorrelation with index option prices. Other studies have documented how contingent claims can be used to elicit valuable forward-looking information about the market's expected spot returns (Ross, 2015; Borovička, Hansen, and Scheinkman, 2016; Schneider and Trojani, 2019; Jensen, Lando, and Pedersen, 2019; Heston, 2021; Bakshi, Gao, and Xue, 2022). But these authors do not study forward returns.

The rest of the paper is organized as follows. Section 2 introduces the VIX-approach, followed by the empirical results and applications in Section 3. We present the VIX-Gamma approach and its application in Section 4. Section 5 concludes. All technical proofs are presented in the Appendixes. In addition, in the Internet Appendix of this paper, we offer extensions of the theory and present additional empirical results.

2 Theory

For a discrete time subscript t, S_t denotes the time-t price of the stock index. $R_{t\to t+1} = \frac{S_{t+1}}{S_t}$ is the gross market return over the time period from t to the next time $t+1$, and $R_{f,t\to t+1}$ is the gross risk-free return over the same time period. We denote the real-world probability measure by $\mathbb{P}$, and the information set at time t by $\mathscr{F}_t$. Let (M_t) be a pricing kernel process and $m_{t,t+1} = \frac{M_{t+1}}{M_t}$ be the stochastic discount factor (SDF) over the period from t to $t+1$. Consequently, the risk-neutral probability measure $\mathbb{Q}$ satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathscr{F}_{t+1}} = R_{f,t\to t+1} m_{t,t+1}.$$

For any $f(S_{t+1}) \in \mathscr{F}_{t+1}$ with suitable integrability condition, its conditional expectation under $\mathbb{P}$ is given by

$$\mathbb{E}_t^{\mathbb{P}}[f(S_{t+1})] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}} f(S_{t+1})\right] = \frac{1}{R_{f,t\to t+1}} \mathbb{E}_t^{\mathbb{Q}}\left[\frac{f(S_{t+1})}{m_{t,t+1}}\right]. \tag{1}$$

Equation (1) states that a conditional expectation of $f(S_{t+1})$ under the real-world probability measure $\mathbb{P}$ is the no-arbitrage time-t price of a contingent claim with payoff $\frac{f(S_{t+1})}{m_{t,t+1}}$ at time $t+1$.[4] We use the notation $\mathbb{E}_t^{\mathbb{P}}[\cdot]$ to highlight the fact that those quantities are under the real-world probability measure. Henceforth, we drop the superscript and use $\mathbb{E}_t[\cdot]$ to denote conditional expectation under the $\mathbb{P}$-measure.

Consider a log-utility-based SDF such that,

$$m_{t,t+1} = \left(\frac{S_t}{S_{t+1}}\right).$$

[4] Bakshi, Gao, and Xue (2022) refer to this known result as the *inverting the Girsanov theorem*.

By Equation (1), we obtain

$$\mathbb{E}_t[R^n_{t\to t+1}] = \frac{1}{R_{f,t\to t+1}}\mathbb{E}^{\mathbb{Q}}_t\left[\left(\frac{S_{t+1}}{S_t}\right)^{n+1}\right], \tag{2}$$

where the right-hand side of the above equation can be synthesized in terms of time-t prices of index call options $C_{t\to t+1}(K)$ that expire at $t+1$. Precisely,

$$\mathbb{E}_t[R^n_{t\to t+1}] = \frac{(n+1)n}{S^{n+1}_t}\int_0^\infty K^{n-1}C_{t\to t+1}(K)dK. \tag{3}$$

This equation for the expected spot return (and higher moments) in terms of options is well studied in the literature (see, e.g. Bakshi, Kapadia, and Madan, 2003; Bakshi and Madan, 2000; Carr and Madan, 1999; Carr, Ellis, and Gupta, 1998; Martin, 2017; Martin and Wagner, 2019). We next move to the forward return (i.e., conditional expected future return).

2.1 VIX-approach

By Equation (1), the forward return is written as[5]

$$\mathbb{E}_t[R_{t+1\to t+2}] = \frac{1}{R_{f,t\to t+2}}\mathbb{E}^{\mathbb{Q}}_t\left[\frac{R_{t+1\to t+2}}{m_{t,t+2}}\right]. \tag{4}$$

Rewriting the right-hand side, we obtain

$$\begin{aligned}
\mathbb{E}_t[R_{t+1\to t+2}] &= \frac{1}{R_{f,t\to t+2}}\mathbb{E}^{\mathbb{Q}}_t\left[\left(\frac{S_{t+2}}{S_{t+1}}\right)\times\left(\frac{S_{t+2}}{S_t}\right)\right] \\
&= \frac{1}{R_{f,t\to t+2}}\mathbb{E}^{\mathbb{Q}}_t\left[(R_{t+1\to t+2})^2\times R_{t\to t+1}\right], \\
&= \frac{1}{R_{f,t\to t+2}}\left\{\mathbb{E}^{\mathbb{Q}}_t\left[(R_{t+1\to t+2})^2\right]\times\mathbb{E}^{Q}_t[R_{t\to t+1}] + Cov^{Q}_t\left((R_{t+1\to t+2})^2, R_{t\to t+1}\right)\right\}.
\end{aligned}$$

[5]For simplicity we only derive the result for $\mathbb{E}_t[R_{t+1\to t+2}]$. The expression of $\mathbb{E}_t[R_{t+n\to t+n+s}]$ is similar and given in the Appendix.

Then,

$$\mathbb{E}_t[R_{t+1\to t+2}] = \frac{1}{R_{f,t+1\to t+2}}\mathbb{E}_t^{\mathbb{Q}}[(R_{t+1\to t+2})^2] + \frac{\theta_t}{R_{f,t\to t+2}}\sqrt{Var_t^{\mathbb{Q}}(R_{t+1\to t+2}^2)}\sqrt{Var_t^{\mathbb{Q}}(R_{t\to t+1})}, \tag{5}$$

where $\theta_t \equiv corr_t^{\mathbb{Q}}\left(R_{t\to t+1}, R_{t+1\to t+2}^2\right)$ is the correlation coefficient between the spot return, $R_{t\to t+1}$, and the future return square, $R_{t+1\to t+2}^2$, under the risk-neutral probability measure $\mathbb{Q}$. In Eq. (5), the risk-neutral conditional variance of $R_{t\to t+1}$ is computed from Eqs. (2) - (3) using index options. The other two terms are the risk-neutral conditional (upon at time t) moment, $\mathbb{E}_t^{\mathbb{Q}}[(R_{t+1\to t+2})^2]$, and the risk-neutral conditional variance of $R_{t+1\to t+2}^2$, which are discussed in the following result.

Proposition 2.1. *Suppose that interest rates are deterministic. For simplicity, let $R = R_{t+1\to t+2}$, $R_f = R_{f,t+1\to t+2}$. Let $F_t = FVIX_{t,t+1\to t+2}$ be the futures price of the VIX index, and σ_t be the implied volatility of at-the-money options on the VIX index. Then $\mathbb{E}_t^{\mathbb{Q}}[R^n]$ can be solved recursively for $n = 2,3,4,\cdots$ via the following approximation formulas,*

$$F_t^2(1+\sigma_t^2) \sim \left(\mathbb{E}_t^{\mathbb{Q}}\left[\left(\frac{R}{R_f}\right)^2\right] - 1\right), \tag{6}$$

$$\frac{1}{2}F_t^2\left(1+\sigma_t^2\right) \sim \sum_{i=1}^{n}(-1)^i\frac{1}{i}\mathbb{E}_t^{\mathbb{Q}}\left[\left(\frac{R}{R_f}-1\right)^i\right], n \geqslant 3. \tag{7}$$

Proof. See Appendix A. □

Proposition 2.1 states that all risk-neutral higher moments of $R_{t+1\to t+2}$ can be computed from the publicly available VIX index, VIX futures, and VIX options.[6] For instance, the risk-neutral conditional moment of $R_{t+1\to t+2}$ is given by

$$\mathbb{E}_t^{\mathbb{Q}}[(R_{t+1\to t+2})^2] \sim R_{f,t+1\to t+2}^2\left(1+F_t^2(1+\sigma_t^2)\right). \tag{8}$$

This formula can be understood as follows. The CBOE's VIX index measures the risk-neutral

[6] Although Proposition 2.1 is given as an approximation, we show that the approximation error is very small for the empirical application in the Appendix.

entropy,

$$VIX^2_{t\to t+1} = \frac{2}{T} L^{\mathbb{Q}}_t \left(\frac{R_{t\to t+1}}{R_{f,t\to t+1}} \right), \tag{9}$$

where $L^{\mathbb{Q}}_t(X) \equiv \log\left[\mathbb{E}^{\mathbb{Q}}_t(X)\right] - \mathbb{E}^{\mathbb{Q}}_t\left[\log(X)\right]$. Among all quadratic polynomials, $\mathbb{E}^{\mathbb{Q}}_t[(R_{t+1\to t+2})^2]$ is the best one (up to a constant) to approximate the risk-neutral conditional entropy of $R_{t+1\to t+2}$, i,e., the second moment of a future VIX. By the equation, $\mathbb{E}^Q[X^2] = \mathbb{E}^Q[X]^2 + Var^Q(X)$, the second moment of a future VIX (represented by a random variable X) is the sum of a square of a *VIX futures* price, $\mathbb{E}^Q[X]$, and a conditional risk-neutral variance of a future VIX. Since the latter can be proxied by the square of the implied volatility of *VIX options*, we use VIX-derivates to compute the risk-neutral conditional second moment of a future return. Similarly, by high-degree polynomial best approximation, we obtain other equations (recursive formula) in Proposition 2.1. Then we can calculate $\mathbb{E}^{\mathbb{Q}}_t[(R_{t+1\to t+2})^3], \mathbb{E}^{\mathbb{Q}}_t[(R_{t+1\to t+2})^4]$, and so on. Accordingly, we calculate $Var^Q_t(R^2_{t+1\to t+2})$ using VIX-derivatives.

Remark 2.1. *Following Martin (2013), the risk-neutral conditional cumulant-generating function $K(\lambda)$ of the relative future return $\frac{R}{R_f}$ is $K(\lambda) = \log\left(\mathbb{E}^{\mathbb{Q}}_t\left[e^{\lambda R/R_f}\right]\right)$. Notice that $\mathbb{E}^{\mathbb{Q}}_t\left[\frac{R}{R_f}\right] = \mathbb{E}^{\mathbb{Q}}_t\left[\mathbb{E}^Q_{t+T}\left(\frac{R}{R_f}\right)\right] = 1$, then*

$$K(\lambda) = \log\left(1 + \lambda + \sum_{n=2}^{\infty} \frac{1}{n!} \mathbb{E}^{\mathbb{Q}}_t\left[\left(\frac{R}{R_f}\right)^n\right] \lambda^n\right).$$

By Proposition 2.1, the function $K(\lambda)$ can be computed from VIX-derivatives.

Now, in computing the forward return in Equation (5), the last ingredient is θ_t. Theoretically speaking, θ_t can be obtained from the risk-neutral bivariate distribution of $(R_{t\to t+1}, R_{t+1\to t+2})$. Thus basket or correlation options can be used to recover the value of this parameter.[7] Nevertheless, there are no available real-time basket or correlation options yet in the financial market to derive the value of θ_t.

[7]For this particular case, Martin (2021) reduces it to be a forward-start option valuation. In general, if $\mathscr{F}_{t+T}$ is generated by $S_1, \cdots, S_{t+T}$, then all conditional information at time t should be recovered by the time t value of some options such as basket options with state variables $S_{t+1}, \cdots, S_{t+T}$. The theory is developed in Tian (2014, 2019) by using the universal approximation theorem from neural networks. Carr and Laurence (2011) also develop a theory in terms of basket options based on random transformation.

In this paper, we suggest two methods to estimate the value of θ_t. The first method is as follows. While θ_t is unknown, we know that the risk-neutral correlation coefficient, $corr_t^{\mathbb{Q}}\left(R^2_{t\to t+1}, R_{t+1\to t+2}\right) = 0$, and in general, $corr_t^{\mathbb{Q}}\left(g(R_{t\to t+1}), R_{t+1\to t+2}\right) = 0$, for any function $g(\cdot)$. This implies that, *to a certain extent*, $R_{t+1\to t+2}$ is independent from $R_{t\to t+1}$. Therefore, it is reasonable to expect that θ_t is close to zero.[8] Indeed, in Section 4 below, we introduce an alternative expression of forward returns in terms of index options and option greeks. As will be shown in Section 4, we can use historical option data to estimate θ_t (the second method), and the averages are fairly close to zero. For these reasons, we first assume that $\theta_t = 0$; thus, the forward return is

$$\mathbb{E}_t[R_{t+1\to t+2}] = \frac{1}{R_{f,t+1\to t+2}}\mathbb{E}_t^Q[R^2_{t+1\to t+2}] \sim R_{f,t+1\to t+2}\left(1 + F_t^2(1+\sigma_t^2)\right). \tag{10}$$

Since VIX-derivatives data alone are sufficient to compute the forward return in the last equation, we call this approach the *VIX-approach* for estimating the forward return.

Remark 2.2. *Similar to Equation (3), in which the conditional moments of spot return can be implied by index options, we can also express the higher moments of future return with VIX-derivatives. Precisely, by assuming* $corr_t^Q\left(R_{t\to t+1}, (R_{t+1\to t+2})^k\right) = 0, k \geqslant 2$, *we have*

$$\mathbb{E}_t\left[(R_{t+1\to t+2})^k\right] = \frac{1}{R_{f,t+1\to t+2}}\mathbb{E}_t^Q\left[(R_{t+1\to t+2})^{k+1}\right]. \tag{11}$$

Thus far, we assume a log-utility-based specification of the pricing kernel process. It can easily be extended to a power specification for a representative CRRA-type agent with a coefficient of constant relative risk aversion γ,

$$m_{t,t+1} = \left(\frac{S_t}{S_{t+1}}\right)^{\gamma}, \quad \gamma \geqslant 1.$$

[8] For instance, if Stein's lemma holds for the risk-neutral bivariate distribution of $(R_{t\to t+1}, R_{t+1\to t+2})$, then $\theta_t = 0$. In general, however, the bivariate distribution of (X,Y) has a rich structure, yielding non zero correlation coefficient between X and Y^2, but zero-correlation between X^2 and Y. We will discuss this issue in Appendix C.

In the Internet Appendix, we present a general expression of forward returns for a power-utility based specification of the stochastic discount factor process. We introduce the concept of power-VIX (PVIX) to express the forward returns and all higher moments of future return in the VIX-approach. We demonstrate that the effect of the risk-aversion parameter γ is marginal. Moreover, we show that both the VIX and VIX-Gamma approachs can be applied to a general class of stochastic discount factor process of the form $m_{t,t+1} = f_t(R_{t\to t+1})$ with a time-dependent smooth function $f_t(x), \forall t$.

3 Empirical Results and Applications

In this section, we use the theoretical results of Section 2 in three novel applications. We start with the first application by applying the VIX-approach to derive the term structure of forward one-month returns. Next, we compute the expected spot return (and higher orders) from index options. Combined with the moments of future return from VIX-derivatives, we recover the market autocorrelation on a real-time basis. Finally, we show how the real-time market autocorrelation can be used to construct a market timing strategy that outperforms the buy and hold benchmark. A common theme of these applications is that the recovered future return and autocorrelation contain valuable forward-looking information not captured by historical measures.

3.1 Data

We collect data for S&P 500 index options and VIX options from OptionMetrics, and obtain VIX index and VIX futures data from the CBOE. On each trading day, we follow Hu and Jacobs (2020) to use linear interpolation to compute daily VIX futures prices, $\text{FVIX}_{t,t+T_1\to t+T_1+T_2}$, with constant maturities for T_1 = 1, 2, 3, 4, 6, and 9 months.[9] Since both VIX index and VIX futures

[9] CFE may list futures for up to nine near-term serial months, as well as five months on the February quarterly cycle associated with the March quarterly cycle for options on S&P 500 (Mencia and Sentana, 2013). We thus choose the maximum constant maturity to be nine months. VIX futures expiration calendar can be found at https://www.macroption.com/vix-expiration-calendar/.

measure the forward-looking implied index volatility over 30 days, T_2 always represents one month.

Panels A and B in Table 1 report summary statistics for VIX index and VIX futures prices. Typically, the VIX futures market is in contango. That is, on average, VIX futures prices are higher than the VIX index, reflecting the volatility risk premium paid by holders of long volatility positions. Panel C reports summary statistics for implied volatility of at-the-money VIX call and put options. After applying standard filters and merging data from different databases, we end up with a sample of daily observations from February 24, 2006 to December 31, 2019. All results are annualized.[10]

3.2 Term structure of forward returns

The expected excess return on the market, or expected equity risk premium, is one of the central quantities of interest in finance and macroeconomics (Martin, 2017; Rapach and Zhou, 2022). In this subsection, we study the shapes of the term structure of forward returns and equity forward risk premiums, as the first application.

To be specific, let

$$f_{t,T} = \mathbb{E}_t\left[R_{t+T\to t+T+1}\right], \forall t, \forall T = 0, 1, \cdots, \tag{12}$$

where $f_t \equiv f_{t,0}$ is the conditional expected spot return $\mathbb{E}_t\left[R_{t\to t+1}\right]$. The one-period forward return $f_{t,T}$ forms a term structure of future returns analogous to the term structure of forward rates: Conditional on the time-t, $f_{t,T} = \mathbb{E}_t\left[\mathbb{E}_{t+T}\left[R_{t+T\to t+T+1}\right]\right] = \mathbb{E}_t\left[f_{t+T}\right]$, which is similar to the equation that the implied forward rate equals the expected spot rate in the fixed-income market.

However, the relationship between equity spot and forward returns is fundamentally different than the relationship between bond spot and forward returns. To see this, suppose $\hat{r}_{t\to t+n}$ is a default-free continuously compounded spot interest rate in effect from time t until the future time $t+n$.[11] In other words, $R_{f,t\to t+n} = \exp(n\hat{r}_{t\to t+n}) = 1/P(t,t+n)$, where $P(t,t+n)$ is the time t

[10] We present full details of the procedure and statistics in the Internet Appendix.
[11] A similar argument also holds for any discrete compounding convention.

price of a default-free zero coupon bond paying 1 at time $t+n$. By the (continuously compounded) yield curve at time t, we mean the mapping $n \to \hat{r}_{t\to t+n}$. At time t, the return that will be realized from holding the zero-coupon bond until maturity at time $t+n$ is *known*. On the other hand, we can also construct a term structure of equity spot returns, say, $n \to \mathbb{E}_t[R_{t\to t+n}]$. However, the realized return $R_{t\to t+n}$ on the equity index is *unknown* at time t, as the expected spot return is distinct from the realized return due to the perpetual nature of equity claims.

Now let $\hat{f}_{t,t+n\to t+n+m}$ be the implied (continuously compounded) forward rate, at time t, from time $t+n$ to $t+n+m$. The well-known relationship between forward and spot rates,

$$\hat{f}_{t,t+n\to t+n+m} = \frac{(m+n)\hat{r}_{t\to t+n+m} - n\hat{r}_{t\to t+n}}{m}, \tag{13}$$

is a straightforward consequence of the no-arbitrage principle. However, the forward equity return, $\mathbb{E}_t[R_{t+n\to t+n+m}]$, cannot be derived similarly in terms of expected spot returns. In fact, we can determine the time-t fair rate, K, of a forward contract maturing at time $t+n$, to exchange the realized return $R_{t+n\to t+n+m}$ at time $t+n+m$. By the no-arbitrage asset pricing principle,

$$K = \mathbb{E}_t^Q[R_{t+n\to t+n+m}] = \mathbb{E}_t^Q\left[\mathbb{E}_{t+n}^Q[R_{t+n\to t+n+m}]\right] = \mathbb{E}_t^Q[R_{f,t+n\to t+n+m}]. \tag{14}$$

Hence, the “implied” forward return on the equity index is the risk-free forward return.

The difference between $\mathbb{E}_t[R_{t+n\to t+n+m}]$ and $\mathbb{E}_t^Q[R_{t+n\to t+n+m}]$ is called the time-t conditional expected forward risk premium. Compared to the equity index’s forward return, the risk-free return in each short time period is small and relatively stable. Therefore, the term structure of $f_{t,T}$ is essentially comparable (in shape) to the term structure of *expected forward risk premiums*, $f_{t,T} - \mathbb{E}_t^Q[R_{f,t+T\to t+T+1}]$, which reduces to $f_{t,T} - R_{f,t+T\to t+T+1}$, provided interest rates are deterministic. Thus, we obtain the term structure of forward one-period returns and expected forward risk premiums by the VIX-approach in Equation (10) and Proposition 2.1.

Table 2 reports the summary statistics for the T-forward one-month returns. We choose T to be 1, 2, 3, 4, 6, and 9 months in $f_{t,T}$. Panel A considers the full sample period from February

24, 2006 to December 31, 2019. The term structure of forward one-month equity returns (and equity risk premiums) is mainly upward-sloping in normal times. On average, the forward one-month return and expected risk premium *increase* with respect to maturity T, except for a slightly downward/flat feature when $T = 6$ months. We next examine the shape of the term structure $f_{t,T}$, for $T = 1,2,3,4,6,9$, when the market is in bad (good) times. Precisely, we use the NBER recessions period, January 1, 2008−June 30, 2009 to represent bad times, and the post-NBER recessions, July 1, 2009−December 31, 2019, to represent good times. As shown in Panel B of Table 2, the term structure of forward one-month returns (and equity forward risk premiums) is *downward* sloping on average in bad times. In contrast, Panel C reveals an *upward*-sloping term structure of one-month future returns (and expected risk premiums), on average, in good times.

Collectively, Figure 1 illustrates the term structure of future one-month returns on average, in "good", "bad", or "overall" times, respectively. The slope of the term structure is pro-cyclical. Furthermore, Figure 2 displays the term structure of future one-month returns for all time t during the NBER recessions. More specifically, we divide the sample period into four shorter ones: January 2008−October 2008; November 2008−January 2009; February 2009−April 2009; and May 2009−June 2009. We observe that the downward-sloping term structure is significantly steep between October 2008 and April 2009 (the most severe period of the 2008/09 global financial crisis). By contrast, Figure 3 shows an upward-sloping term structure of forward one-month returns (expected risk premiums) most of the time between 2009 and 2019.

It is interesting to compare our results on the equity forward term structure with recent studies of the equity term structure in the literature (see, for instance, Binsbergen, Brandt, and Koije, 2012; Binsbergen and Koijen, 2017; Gormsen, 2021). By a term structure of equity risk premia, these previous studies refer to the relationship between a one-period spot return premia of maturing asset with the maturity.[12] Specifically, a zero-coupon equity or dividend strip is a claim with only one dividend payment at a future time, analogous to a zero-coupon bond. Let $P_{n,t}$ be the price at time t of a claim (dividend strip) to the dividend at time $t+n$. Then, the time-t price of the underlying

[12] In Chabi-Yo and Loudis (2020), a term structure is a lower bound of hold-to-maturity expected spot returns at various horizons. The authors show that the term structure of the (lower bound) of spot returns is downward-sloping during turbulent times but upward-sloping during normal times.

index is $S_t = \sum_{n=1}^{\infty} P_{n,t}$. Let $R_{n,t}$ be the one-period spot return of the dividend strip maturing $t+n$ from period t to $t+1$. That is,

$$R_{n,t+1} = \frac{P_{n-1,t+1}}{P_{n,t}}. \tag{15}$$

The spot return of the underlying index is

$$R_{t\to t+1} = \sum_{n=1}^{\infty} \omega_{n,t} R_{n,t+1}, \omega_{n,t} = \frac{P_{n,t}}{S_t}. \tag{16}$$

Then, the term structure of the dividend return, $n \to R_{n,t+1}$, illuminates the contribution of the dividend return $R_{n,t+1}$ to the spot return $R_{t\to t+1}$. [13] For example, an upward-sloping term structure of the dividend return shows a higher contribution of the dividend strip maturing $t+n$ to the underlying aggregate market index return $R_{t\to t+1}$ when the maturity n is higher, and vice versa. In contrast, we focus on the term structure of the aggregate equity market's forward returns, $T \to R_{t+T\to t+T+1}$, a relationship between T and the forward return starting from period $t+T$ to $t+T+1$.

In this regard, we document a new stylized fact that the term structures of forward one-period returns and expected forward risk premiums implied by derivatives markets are *pro-cyclical*. The pro-cyclicality can be explained as follows. By Proposition 2.1, the conditional expected future one-month returns in T months are essentially determined by the futures prices of VIX over the same period. Therefore, a pro-cyclical term structure of equity risk premia is consistent with Hu and Jacobs (2020) which documents that VIX futures prices tend to have an upward sloping term structure during normal times and tend to become inverted or hump-shaped in times of market turbulence.

[13] Gormsen (2021); Bansal, Miller, Song, and Yaron (2021) study the term structure of the dividend future return, or $\theta^{n,m} = \mathbb{E}_t[R_{n,t} - \mathbb{E}_{m,t}]$ for long maturity n and short maturity $m < n$. Similarly, Binsbergen, Brandt, and Koije (2012); Binsbergen and Koijen (2017) consider the difference between short-term assets with all dividend payments until T, say three years, and long-term assets with all remaining future dividends.

3.3 Market autocorrelation

Building on the theoretical results of Section 2, we turn now to the question of predicting the expected market return with past return observations. Specifically, we are interested in computing the conditional market autocorrelation, $corr_t\left(R_{t\to t+1}, R_{t+1\to t+2}\right)$, under the real-world probability measure. This conditional market autocorrelation reveals how two future returns change from the perspective of time t in two consecutive periods. Our goal is to compute these conditional market autocorrelations from derivatives data.

For this purpose, we compute

$$Cov_t\left(R_{t\to t+T}, R_{t+T\to t+T+1}\right) = \mathbb{E}_t\left[R_{t\to t+T+1}\right] - \mathbb{E}_t\left[R_{t\to t+T}\right] \times \mathbb{E}_t\left[R_{t+T\to t+T+1}\right]. \tag{17}$$

By Equations (3) and (10), the expected spot return can be recovered from equity index option prices, and the expected future return can be obtained from the VIX-derivatives. Similarly, we can estimate $Var_t(R_{t\to t+T}), Var_t(R_{t+T\to t+T+1})$, and then the autocorrelation coefficient $corr_t\left(R_{t\to t+1}, R_{t+1\to t+2}\right)$, under the real-world probability measure.

Table 3 (Panel A) reports the market autocorrelation on the S&P 500 index. We use the average value between the implied volatility of at-the-money put and call VIX options as a proxy for σ_t. Following the VIX futures structure, we consider T_1 to be 1, 2, 3, 4, 6 and 9 months, and T_2 to be fixed for 1 month. Across columns, we observe significantly *negative* coefficients, suggesting a persistent short-term reversal. For instance, when T_1 is 1 month, $corr_t(R_{t\to t+1mo}, R_{t+1mo\to t+2mo})$ is on average -20.90% with a t-stat of 18.10. On average, the market autocorrelation on S&P 500 index is around -20% to -40%. Notably, the numbers in Table 3 are comparable to Chabi-Yo (2019) and Martin (2021). Using index options data, Chabi-Yo (2019) estimates that the upper bounds of autocorrelation vary from -28% to -3%; with price quotes of forward-start options from a major investment bank, Martin (2021) estimates that the autocorrelation of the S&P 500 index is between -20% and -40% for a small number of days. Table 3 suggests that our no-arbitrage framework, in Proposition 2.1, is consistent with the pricing of over-the-counter derivatives in the market.

Figure 4 displays the time-series of $corr_t(R_{t\to t+1mo}, R_{t+1mo\to t+2mo})$. It is well known that the month-month autocorrelation coefficient from the historical data is close to zero ((Lo and MacKinlay, 1988, 1990), and Table 3, Panel B). From the perspective of the derivatives market, however, the autocorrelation coefficients can be either negative or positive, though they are negative most of the time.

As a comparison, we also compute the *month-to-month autocorrelation* between two consecutive calendar months using historical stock return, including January/February, February/March, $\cdots$, and December/January. Figure 5 plots the consecutive month-to-month autocorrelation over various periods with the data prior to January 1927 obtained from Robert Shiller's website. In contrast to virtually zero month-month autocorrelation coefficient in (Lo and MacKinlay, 1988, 1990), the autocorrelation between two consecutive months can be significantly nonzero. It can be either positive or negative, depending on the sample of the data. For example, the autocorrelation of March/April is around 10% over 1871-2019, but -20% over a recent time period 1990-2019.

We next compute the consecutive month-to-month autocorrelation from the derivative market as explained in Section 2. For consistency, we restrict the sample period to $2006-2019$, calculate $corr_t(R_{t\to t+1mo}, R_{t+1mo\to t+2mo})$ on the first day of each month and take the simple average within each of the 12 calendar months of the year. For example, for March/April, we compute the correlation coefficient with VIX-derivatives data on March 1 in each year and then take a simple average. Our results are displayed in Figure 6, in which the solid red line displays the month-to-month autocorrelations from the derivative market. By contrast, the blue dot line represents the month-month autocorrelation from the historical stock return (as in Figure 5). Both methods yield a similar pattern of consecutive month-month autocorrelation, but the derivative approach results more negatively in magnitude. In fact, we demonstrate negative autocorrelation between any two consecutive months from the derivatives market. As an illustration, both yield similar autocorrelation coefficients of -35% between February and March, and -10% between December and January. Between May and June, the derivative approach implies an autocorrelation as large as -40%, while the historical returns suggest a value of -20%. Moreover, the monthly return displays a stronger reversal in specific periods than others (for instance, from February to March, May to June, July to August, and December to January) by both approaches. In summary, the derivative market reveals a robust short-term reversal in the stock market from a forward-looking perspective.

Finally, using the VIX approach, we can derive the conditional correlation coefficient between two spot returns. For instance, in the following conditional correlation between $R_{t\to t+1}$ and $R_{t\to t+2}$,

$$corr_t(R_{t\to t+1}, R_{t\to t+2}) = \frac{\mathbb{E}_t[R_{t\to t+1}R_{t\to t+2}] - \mathbb{E}_t[R_{t\to t+1}]\mathbb{E}_t[R_{t\to t+2}]}{\sqrt{Var_t(R_{t\to t+1})}\sqrt{Var_t(R_{t\to t+2})}}, \tag{18}$$

every term except $\mathbb{E}_t[R_{t\to t+1}R_{t\to t+2}]$ is obtained from index options. Similar to Equation (5), we have

$$\begin{aligned}
\mathbb{E}_t[R_{t\to t+1}R_{t\to t+2}] &= \frac{1}{R_{f,t\to t+2}}\mathbb{E}_t^{\mathbb{Q}}\left[(R_{t+1\to t+2})^2 \times (R_{t\to t+1})^3\right], \\
&= \frac{1}{R_{f,t\to t+2}}\left\{\mathbb{E}_t^{\mathbb{Q}}\left[(R_{t+1\to t+2})^2\right] \times \mathbb{E}_t^{Q}\left[(R_{t\to t+1})^3\right] + Cov_t^{Q}\left((R_{t+1\to t+2})^2, (R_{t\to t+1})^3\right)\right\} \\
&\sim \frac{1}{R_{f,t\to t+2}}\mathbb{E}_t^{\mathbb{Q}}\left[(R_{t+1\to t+2})^2\right] \times \mathbb{E}_t^{Q}\left[(R_{t\to t+1})^3\right],
\end{aligned}$$

assuming $corr_t^{Q}\left((R_{t+1\to t+2})^2, (R_{t\to t+1})^3\right) = 0$.[14] Proposition 2.1 and Equation (3) can be used to derive $\mathbb{E}_t[R_{t\to t+1}R_{t\to t+2}]$ with index options and VIX derivatives.

Panels C and D in Table 3 report this new correlation coefficient calculated using either derivatives or historical stock returns. As shown, using historical data, the autocorrelation between two spot returns is significantly positive. For example, on average, the correlation coefficient between the one-month spot return and the two-month spot return is 0.746. However, based on derivatives market information, the one-month spot return and two-month spot return are more significantly positively correlated (0.83). Furthermore, the market autocorrelation between

[14] By a similar method in Section 4, we can also justify this assumption empirically. Without the VIX approach, we need prices of some generalized forward-start options or spread options which are at present traded only in over-the-counter markets.

two spot returns, for $T_1 = 1,2,3,4,6,9$ months, using derivatives market information is more significant than the correlation coefficients derived from the historical stock returns data. Given the robust short-term reversal from the derivatives (Panel A), we document that the derivatives market information reveals a very substantial and higher correlated movement between two spot returns.

3.4 Market timing

To demonstrate the investment value of the VIX-approach, we next propose a market timing strategy as the third application. We start by constructing a reversal trading signal based on the forward-looking autocorrelation from the derivatives. We then discuss several evaluation criteria and report the out-of-sample performance of the marketing timing strategy.

3.4.1 A reversal trading signal

Motivated by the short-term reversal documented in Table 3, we construct a reversal signal based on both realized cumulative excess returns and the conditional derivative-based autocorrelation. Specifically, we define the reversal signal at time t as,

$$\tilde{S}_{t,K}\left[r_{t-K\to t}, corr_{t-K}(r_{t-K\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-K\to t} > 0 \quad \& \quad corr_{t-K}(r_{t-K\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-K\to t} < 0 \quad \& \quad corr_{t-K}(r_{t-K\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

where $r_{t-K\to t} = R_{t-K\to t} - R_{f,t-K\to t}$ is the realized cumulative excess return over the past K months, $corr_{t-K}(r_{t-K\to t}, r_{t\to t+1})$ is calculated from the derivatives, and K = 1, 2, 3, 4, 6, and 9 months. In total, we have six market reversal signals at time t.[15]

Following the market reversal signal in 19, we trade the market by implementing a zero-cost strategy at the beginning of the subsequent month. As an illustration, if we use the one-month

[15] Notice that computing the autocovariance is sufficient in constructing the reversal signal. We do not necessarily need the autocorrelation in this case.

reversal signal as a trading signal at time t and implement the corresponding market timing strategy, the realized return in the subsequent month is

$$\eta\left[\tilde{S}_{t,1}\right]=\begin{cases} r_{t\to t+1}, & \text{if} \quad \tilde{S}_{t,1}=1, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

We call the strategy based on the 1-month reversal signal the *single timing strategy*. It is also possible to use all six reversal signals together, which we call *combination timing strategy*. For instance, employing the combination strategy would entail being long the market if $\sum_K \tilde{S}_{t,K}$ is greater than some threshold, ξ, an integer ranging from 2 to 5. Following the combination timing strategy, the realized return in the next month is

$$\eta\left[\tilde{S}_{t,K},\forall K;\xi\right]=\begin{cases} r_{t\to t+1}, & \text{if} \quad \sum_K \tilde{S}_{t,K}\geqslant \xi, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

In other words, we should be long the market if and only if at least ξ reversal signals defined in Equation (19) indicate long signals.

3.4.2 Performance evaluation

To evaluate the market timing strategy's performance, we compute four performance measures based on the mean $\hat{\mu}_j$ and standard deviation $\hat{\sigma}_j$ of the out-of-sample realized returns of strategy j.

First, we measure the *out-of-sample Sharpe ratio* (SRatio) and the *certainty-equivalent return* (CEQ) of each strategy,

$$\hat{s}_j=\frac{\hat{\mu}_j}{\hat{\sigma}_j} \tag{22}$$

and

$$C\hat{E}Q_j=\hat{\mu}_j-\frac{\gamma}{2}\hat{\sigma}_j^2, \tag{23}$$

where γ is chosen to be 1, consistent with the log-utility specification of the stochastic discount factor in Section 2. To test whether the Sharpe ratios of two strategies are statistically distinguishable, we follow DeMiguel, Garlappi, and Uppal (2009) to compute the p-value of the difference. We use the approach suggested by Jobson and Korkie (1981) after making the correction pointed out in Memmel (2003). A similar test can be applied to the CEQ difference.[16]

Next, we compute DeMiguel, Garlappi, and Uppal's (2009) *return-loss* with respect to a benchmark. We choose the buy and hold strategy as the benchmark as in Gao, Han, Li, and Zhou (2018). Precisely, if $\{\hat{\mu}_b, \hat{\sigma}_b\}$ are the monthly out-of-sample mean and volatility of the excess returns from the buy-and-hold strategy, the return-loss from strategy j is

$$\text{return-loss}_j = \left(\frac{\hat{\mu}_b}{\hat{\sigma}_b}\right) \times \hat{\sigma}_j - \hat{\mu}_j. \tag{24}$$

In other words, the return-loss is the additional return required in order for the performance of strategy j to be consistent with the performance of the benchmark. A negative value indicates that strategy j outperforms the benchmark as measured by the Sharpe ratio.

Lastly, we calculate the *performance fee* suggested in Fleming, Kirby, and Ostdiek (2001), defined as the maximum fee that a quadratic-utility investor would be willing to pay to switch from the benchmark to the timing strategy. This fee is estimated as the value of Δ that solves

$$\sum_t \left[\left(R_{j,t} - \Delta\right) - \frac{\gamma}{2(1+\gamma)} \left(R_{j,t} - \Delta\right)^2 \right] = \sum_t \left[R_{b,t} - \frac{\gamma}{2(1+\gamma)} R_{b,t}^2 \right], \tag{25}$$

where $R_{j,t}$ and $R_{b,t}$ denote the out-of-sample realized returns for timing strategy j and the benchmark, respectively. We report the estimates of Δ in units of basis points per annum.

3.4.3 Out-of-sample performance

Panel A of Table 4 reports the performance measured on (annualized) returns generated from the market timing strategies over the full sample period. The market timing strategy delivers good

[16] DeMiguel, Garlappi, and Uppal (2009), on pages 1928–1929, provide a detailed description of how to construct p-values foe differences in Sharpe ratios.

realized returns, but does not necessarily lead to the highest realized return on average. The average realized return is about 4.433% *per annum* by the single timing strategy, whereas the average return from buy and hold is about 5.489%. This is reasonable given the market's upward trend from 2006 to 2019, regardless of the financial crisis around 2008 or a market downturn in 2018. More important questions to ask are: (i) whether the market timing strategy delivers a higher Sharpe ratio; and (ii) whether it "predicts" bad market times.

For the first question, all timing strategies produce minor standard deviations than the benchmark, suggesting that the market timing strategy is more conservative than the benchmark. For instance, the standard deviation is 9.616% per annum for the single timing strategy; but 14.789% for buy and hold, which is almost twice large. As a result, the single timing strategy produces a Sharpe ratio of 0.461, whereas the buy and hold only achieves 0.371. We also see that the last combination timing strategy delivers a higher Sharpe ratio of 0.467.

For the second question, we evaluate the out-of-sample performance during the NBER recessions in Panel B of Table 4. Not surprisingly, the buy and hold strategy suffers a dramatic loss, yielding a negative average return of −32.304% per annum, along with a standard deviation as high as 25.565%, during January 2008−June 2009. Consequently, the Sharpe ratio of the benchmark is around −1.264. In contrast, the single timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$, achieves an average return of −7.027% per annum, along with a much smaller standard deviation of 14.420%. Although the Sharpe ratio from the single timing strategy is also negative, around −0.487, it exhibits a significant economic value relative to the benchmark, as suggested by the return-loss and the performance fee. The −11.194% return-loss value of the single timing strategy suggests that investors are willing to pay as high as 11% per annum to switch from the buy and hold to the market timing strategy. Likewise, the quadratic-utility investor would be willing to pay an estimated 2630 basis points annually to switch from the benchmark portfolio to the single timing strategy. Remarkably and consistently, during the market crisis, all single and combination timing strategies yield higher average returns, smaller standard deviations, higher Sharpe ratios, larger CEQs, negative return-loss measures, and positive performance fees than the buy and hold strategy.

Furthermore, we plot the realized returns generated from the single market timing strategy and the buy and hold strategy during the NBER recessions from January 2008 to June 2009 in Figure 7. This recession period overlaps the 2008/09 global financial crisis. We observe that the market timing strategy based on a one-month reversal signal avoids significant market crashes in January, June, September, October of 2008, and January of 2009. To summarize, we show that the robust short-term reversal identified by the derivative market does reveal valuable information on future market downturns, and the associated economic value can be substantial.

4 VIX-Gamma approach

So far, we have assumed that $\theta_t = corr_t^{\mathbb{Q}}\left(R_{t\to t+1}, R^2_{t+1\to t+2}\right) = 0$. One appealing feature of the VIX-approach is that it provides an effective way to compute forward return in real-time. However, we need to justify this assumption for the equity index, as this zero risk-neutral correlation coefficient assumption fails for a general risk-neutral bivariate distribution in a no-arbitrage pricing model.[17] Consequently, we turn our attention now to the problem of estimation and prediction of θ_t using available derivatives data.

In this section, we first provide an alternative expression of forward return in terms of option prices and gammas. Although this formula cannot be used in real-time to compute forward return (see explanations below), we can combine this expression and Equation (5) to construct a predictor for the parameter θ_t using available historical index options and VIX-derivatives. Then, we use this estimation of θ_t at time t to compute the forward return. We call this methodology, the VIX-Gamma approach, and apply the VIX-Gamma approach to the market timing strategy on a real-time basis.

[17] We provide a simple example in Appendix C to illustrate that the risk-neutral correlation coefficient can be any nonzero number between -1 and 1 in a simple two-period no-arbitrage asset pricing model.

4.1 Forward return from index option market

In this subsection, we provide an alternative expression of forward return from the options market.

Proposition 4.1. *Suppose that interest rates are deterministic. Then,*

$$\mathbb{E}_t[R_{t+1\to t+2}] = 2S_t \int_0^\infty \frac{C_t''(S_t, L)}{L^3} \left(\underbrace{\int_0^\infty C_{t+1}(L, K)\, dK}_{\text{inside-integral, } I_{t+1}(L)\text{, known at } t} \right) dL, \tag{26}$$

where

- S_t = *underlying index price observed at time* t*;*
- $C_t''(S, L)$ = *the call option gamma at time* t *with the underlying* S_t *and strike price* L*;*
- $C_{t+1}(L, K)$ = *the call option price at time* $t+1$ *with the underlying* L *and strike price* K*.*

Proof. See Appendix B □

Compared with Eq. (10) for the equity index, Equation (26) presents an alternative formula of forward return for a general underlying variable R_t. Specifically, to compute a forward return at time t, there are two sets of option data required in Equation (26) in addition to the asset price S_t. First, the call option gamma, $C_t''(S, L)$, with the underlying S_t, strike price L, and maturity $t+1$, is needed. The option gamma is available in real-time. Second, the time $t+1$ price of call option price with time to maturity $t+2$, *when the underlying price is* L *at time* $t+1$, from time t perspective. Notice that this price $C_{t+1}(L, K)$ is known at time t, but it is not real-time, since the underlying index only achieves one particular number at time $t+1$. Therefore, we need to explain why the option gamma and $C_{t+1}(L, K)$ are involved in this equation (the details are given in the Appendix).

First of all, the number $C_{t+1}(L, K)$ involved in Equation (26) is well-defined at time t. As a simple illustrative example, the Black Scholes option formula, assuming the underlying asset has a normal distribution with a constant volatility parameter σ, presents a known option price at a future time as follows:

$$C_{t+1}(L,K) = LN(d_1) - \frac{K}{R_{f,t+1\to t+2}}N(d_1 - \sigma\sqrt{\Delta t}), d_1 = \frac{log(L/K) + (r_t + \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}.$$

When we write $C_{t+1}(S_{t+1},K)$ at time t, the reason we do not know the option price precisely is that we do not know the realized value S_{t+1}. However, the no-arbitrage asset pricing theory guarantees the **relationship** between $C_{t+1}(S_{t+1},K)$ and S_{t+1}. In particular, when S_{t+1} achieves a number L, the price $C_{t+1}(L,K)$ is known. This argument holds in general, regardless of the distribution of S_{t+1}. The reason is simple. Given the specification of the stochastic discount factor, we know at time t a precise relationship between the underling index price S_{t+1} and the option price $C_{t+1}(S_{t+1},K)$ for any conditional distribution of S_{t+1}. Hence, $C_{t+1}(L,K)$ is well-defined at time t.

Second, even though $C_{t+1}(L,K)$ is known at time t, its expression could be complicated. Under the power-specification of the stochastic discount factor, we obtain

$$\mathbb{E}_t[R_{t+1\to t+2}] = \frac{2}{R_{f,t\to t+1}S_t}\int_0^{\infty}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{C_{t+1}(S_{t+1},K)}{S_{t+1}}\right]dK, \tag{27}$$

which resembles a similar insight in Equation (3) for a future return (a proof is given in the Appendix). If the specification of the stochastic discount factor contains a volatility component, as discussed in Bakshi, Crosby, Gao, and Zhou (2021); Babaoğlu, Christoffersen, Heston, and Jacobs (2018); Christoffersen, Heston, and Jacobs (2013), the above expression of the forward return would be different, since $C_{t+1}(L,K)$ would also depend on the volatility at time t. But, such a specification involving a volatility component is beyond the scope of the present paper.[18]

Third, why do we need option gammas in Equation (26) for a future return, whereas only option prices are required to compute expected spot return $\mathbb{E}_t[R_{t\to t+1}]$ in Equation (3)? This difference seems substantial since $R_{t+1\to t+2}$ is just the index return in a future time period $[t+1,t+2]$. We

[18]As shown in the Online Appendix, a general class of path-independent specification of the stochastic discount factor also implies the forward returns with the derivatives using the same approach. In this paper, we focus on a power-specification of the stochastic discount factor to derive the forward returns from derivatives and document its asset pricing implications.

notice that the term inside the integral is time t price (ignoring the interest rate) of a future payoff $\frac{C_{t+1}(S_{t+1},K)}{S_{t+1}}$ at time $t+1$; thus, it is essentially a forward-start option price with payoff $\frac{\max(S_{t+2}-K,0)}{S_{t+1}}$ at time $t+2$. We write $\Pi(L|S_t)$ be the conditional distribution of S_{t+1}. Namely,

$$\Pi(L|S_t) = \int_0^L q(z|S_t)dz$$

where $q(z|S_t)$ is the conditional density function under the risk-neutral probability measure. It is known in the options literature that

$$d\Pi(L|S_t) = R_{f,t\to t+1}\frac{\partial^2 C_t(S_t,L)}{\partial L^2}dL$$

Therefore, we can represent the forward-start option at time t as

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{C_{t+1}(S_{t+1},K)}{S_{t+1}}\right] = \int_0^\infty \frac{C_{t+1}(L,K)}{L}R_{f,t\to t+1}\frac{\partial^2 C_t(S_t,L)}{\partial L^2}dL.$$

Finally, Equation (26) follows from the following relationship between option gamma and strike-gamma as follows.

$$L^2\frac{\partial^2 C_t(S_t,L)}{\partial L^2} = S_t^2\frac{\partial^2 C_t(S_t,L)}{\partial S_t^2},$$

in which we can use the option gamma to replace the strike gamma up to a constant.

Remark 4.1. *Similarly, we can derive $\mathbb{E}_t\left[R^k_{t+1\to t+2}\right], k \geqslant 2$ as follows.*

$$\mathbb{E}_t\left[R^k_{t+1\to t+2}\right] = (k+1)kS_t\int_0^\infty \frac{C_t''(S_t,L)}{L^{k+2}}\left(\underbrace{\int_0^\infty K^{k-1}C_{t+1}(L,K)\,dK}\right)dL. \tag{28}$$

Therefore, we can obtain the conditional distribution of a future return $R_{t+1\to t+2}$ in terms of index option prices and index gamma.

It is worth mentioning that Proposition 4.1 cannot be used directly since $C_{t+1}(L,K)$ cannot be calculated *precisely*, at time t, since we do not specify the conditional distribution of S_{t+1}. For this reason, there is no way to find such data at time t to derive the forward return in real-time. This limitation of Proposition 4.1 prevents us from deriving the forward return from the option market, compared with the real-time VIX-approach. In the next section, we explain how to combine both Proposition 4.1 and Eq. (10) to introduce an improved VIX-Gamma approach for the forward return.

4.2 VIX-Gamma approach

Suppose our objective is to estimate the number θ_t at time t. One procedure is as follows.

First, at time $t-1$, we calculate all risk-neutral return quantities on the right-hand side of Equation (5) by Proposition 2.1, except for θ_{t-1}. Second, assuming the index price S_t is realized at time t. By the homogeneous property of the option price, $C_t(L,K) = C_t(S_t, S_t K/L)\frac{L}{S_t}$, we are able to calculate $C_t(L,K)$ at time t for any L. Therefore, at time $t-1$, we compute the left-hand side, $\mathbb{E}_{t-1}[R_{t\to t+1}]$, of Equation (5) directly by Proposition 4.1. By equating the left-hand and right-hand sides calculations, we calculate the value of θ_{t-1}. Figure 8 provides a visual illustration on how to understand Equation (26). Finally, at time t, we compute the forward return by,

$$\mathbb{E}_t[R_{t+1\to t+2}] = \frac{1}{R_{f,t+1\to t+2}}\mathbb{E}_t^Q[R_{t+1\to t+2}] + \frac{\hat{\theta}_t}{R_{f,t\to t+2}}\sqrt{Var_t^Q(R_{t\to t+1})}\sqrt{Var_t^Q(R_{t+1\to t+2}^2)}, \quad (29)$$

where $\hat{\theta}_t = \theta_{t-1}$,

In the above procedure, we use the risk-neutral correlation coefficient θ_{t-1} as a predictor of θ_t. The reason is straightforward. From an econometrics perspective, we can investigate the time-series property of $\theta_s, s < t$, at time t. Then, we can use statistics of this time-series $\{\theta_s, s < t\}$ to predict θ_t by an econometrics study. For example, if this time series is stationary, θ_{t-1} is a good indicator of θ_t. More generally, if this time series is ergodic, the sample average of $\theta_s, s < t$ is a good indicator of θ_t. A VIX-Gamma approach is to use a predictor $\hat{\theta}_t$ from the statistics of time-series of all past θ's in Equation (29).

Table 5 reports some statistics of the time series of θ_t that is calculated with available derivative data. We find out the sample average of θ_t is relatively stable across different periods and close to zero. For instance, the average value of θ_t in different periods varies between -0.092 to 0.049 except for the calendar year 2010−2011. Moreover, the moving average of θ_t from 6 months to 5 years belongs to [-0.046, 0.013]. And for the entire period, the moving average of θ_t is about -0.06. Therefore, we have justified the VIX-approach by assuming a zero value of θ_t for the equity index.

The difference between the VIX-Gamma approach, Equations (29), and the VIX-approach is a non-zero predictor $\hat{\theta}_t$, which relies on all index (prices and gammas) up to (and include) time t, and VIX derivatives data prior to time t. In the end, we make use of all available VIX derivative data. all option price and option gamma data until to time t, to calculate the forward return $\mathbb{E}_t[R_{t+1\to t+2}]$. Since the VIX-Gamma approach relies on all historical and current derivatives data, it provides a *forward-backward* perspective of future returns. In contrast, the VIX approach is *forward-looking* because current VX-derivatives data are required.

4.3 Market timing by VIX-Gamma

In this subsection, we implement the same market timing strategy with the VIX-Gamma approach. Same as before, we compute the market autocovariance (autocorrelation) on the market and construct the market timing strategy. For brevity, we simply choose the single timing strategy where we rely on the 1-month reversal signal only. We use the θ_t that are predicted recursively as explained in the last subsection.

Table 5 reports several key out-of-sample performance measures of VIX-Gamma approach and the buy and hold benchmark. Same as in Table 4, we consider both full sample period and the NBER recession subperiod during January 2008−June 2009. We observe that, in the full sample, the VIX-Gamma approach outperforms the buy and hold with a higher average return, smaller standard deviation. Remarkably, the Sharpe ratio increases by 35%, from 0.371 by buy and hold to 0.500 by VIX-Gamma. Moreover, the Sharpe ratio difference of 0.129 is statistically significant.

Likewise, the CEQ difference is also significantly positive at the 5% level. The negative return-loss and positive performance fee both suggest that the market timing strategy improves significantly by considering much more options data in VIX-Gamma approach.

A more striking result is given in Panel B of Table 5, during the NBER recessions period. In comparison to the negative mean returns and Sharpe ratios from buy and hold strategy, and those from the VIX-approach based timing strategy in Panel B of Table 4, here we observe a positive average return, a positive Sharpe ratio, and a positive CEQ. Compared with the return-loss of -11.194% and the performance fee of 2,630 basis points for the VIX-approach based single timing strategy in Panel B of Table 4, the two quantities jump to -31.153% and 3,948 basis points, separately, once we switch to the VIX-Gamma approach. On the whole, those results highlight the improved forecasting gains associated with the VIX-Gamma approach, and justify the investment value of studying the conditional expected returns from the derivatives.

The difference between our market timing strategy and the benchmark strategy is that we long the market only when the signal shows a positive market excess return in the following month. In contrast, the benchmark strategy is long the market persistently. In other words, our market timing strategy is to stay away from the stock market if the signal from the derivative market suggests a future market downturn. Therefore, the relative performance of the market timing strategy mainly depends on whether the reversal signal identified indeed reveals valuable information about the market return in the following month.

As shown in Figure 9, the reversal signal from the VIX-Gamma approach more accurately *predicts* the market downturn than the VIX-approach in Figure 7. Remarkably, we find that the trading signals from the VIX-Gamma approach successfully predicted all market crashes during the 2008/09 global financial crisis, except for the most severe one in October 2008. Remarkably, it also captures the upside potentials, for instance, in April and May of 2009, which seems missing in the VIX-approach.

Finally, we plot the realized returns during in full sample period in Figure 10. To highlight the predictive power of the timing strategy, we shadow the area below zero. The reversal signal from

VIX-Gamma does seem to predict the market downturn, particularly when the market crashed in 2008−2009, 2014, 2015, and 2018−2019.

5 Conclusion

In this paper, we express the equity index's forward return by market available derivatives data–index option prices and gammas, VIX-futures, and VIX-option prices. Since this expression depends on all historical derivative and current derivative data, this expression yields a term structure of forward returns from a forward-backward perspective without relying on any model assumption about the equity index.

We present three applications of this expression of forward return from derivatives, including the pro-cyclic term structure of forward returns, robust autocorrelation analysis and short-term reversal pattern, and a profitable dynamic market-timing strategy. The forward return reveals future market drawdowns and captures upward market movements, yielding substantial economic value. Overall, we demonstrate the significance of derivatives market information in estimating expected returns in the future (Miller, 1999, page 100).

Appendix

Appendix A Proof of Proposition 2.1

The proof is divided into several steps.

Step 1. We first derive an approximation formula of VIX as follows

$$VIX^2_{t\to t+T} \sim \frac{1}{T}\left(\mathbb{E}^Q_t\left[\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right)^2\right]-1\right). \tag{A1}$$

By using the second-order expansion of $log(1+x) \sim x - \frac{1}{2}x^2$ when x closes to zero, and $\frac{R_{t\to t+T}}{R_{f,t\to t+T}}$ sufficiently closes to one, we obtain

$$log\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right) \sim \frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1 - \frac{1}{2}\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1\right)^2. \tag{A2}$$

By taking the conditional expectation under the risk-neutral probability measure $\mathbb{Q}$, and using the relation that $\mathbb{E}^{\mathbb{Q}}_t\left[\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right] = 1$, we obtain

$$\mathbb{E}^{\mathbb{Q}}_t \log\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right) \sim \frac{1}{2} - \frac{1}{2}\mathbb{E}^{\mathbb{Q}}_t\left[\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right)^2\right]. \tag{A3}$$

Recall the definition of VIX as a risk-neutral entropy

$$VIX^2_{t\to t+T} = \frac{2}{T}L^{\mathbb{Q}}_t\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right) \tag{A4}$$

where $L^{\mathbb{Q}}_t(X) \equiv \log \mathbb{E}^{\mathbb{Q}}_t X - \mathbb{E}^{\mathbb{Q}}_t \log X$. By Equation (A3), we obtain

$$VIX^2_{t\to t+T} \sim \frac{1}{T}\left(\mathbb{E}^{\mathbb{Q}}_t\left[\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right)^2\right]-1\right), \tag{A5}$$

since $\log \mathbb{E}^{\mathbb{Q}}_t\left[\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right] = 0$.

Step 2. We derive the result for $\mathbb{E}_t^{\mathbb{Q}}[R^n]$ when $n = 2$. We use the formula (A1) for the time period from $t+T_1$ to $t+T_1+T_2$,

$$VIX^2_{t+T_1\to t+T_1+T_2} \sim \frac{1}{T_2}\left(\mathbb{E}^{\mathbb{Q}}_{t+T_1}\left[\left(\frac{R}{R_f}\right)^2\right]-1\right).$$

Here, to simplify notation, we write $R = R_{t+T_1\to t+T_1+T_2}, R_f = R_{f,t+T_1\to t+T_1+T_2}$.

By applying the conditional expectation of the last equation at time t under the $\mathbb{Q}$-measure , we have

$$\mathbb{E}_t^{\mathbb{Q}}[VIX^2_{t+T_1\to t+T_1+T_2}] \sim \frac{1}{T_2}\left(\mathbb{E}^{\mathbb{Q}}_{t}\left[\left(\frac{R}{R_f}\right)^2\right]-1\right), \tag{A6}$$

where the left-hand side can be expressed as

$$\mathbb{E}_t^{\mathbb{Q}}[VIX^2_{t+T_1\to t+T_1+T_2}] = \left(\underbrace{\mathbb{E}_t^{\mathbb{Q}}[VIX_{t+T_1\to t+T_1+T_2}]}_{FVIX_{t,t+T_1\to t+T_1+T_2}}\right)^2 + Var_t^{\mathbb{Q}}(VIX_{t+T_1\to t+T_1+T_2}).$$

Here, the first term on the right-hand side of the last equation is the square of the VIX future by the risk-neutral pricing formula, and the second term is the conditional variance $Var_t^{\mathbb{Q}}(VIX_{t+T_1\to t+T_1+T_2})$.

We now consider the VIX option with maturity $t+T_1$ and the underlying is $VIX_{t+T_1\to t+T_1+T_2}$. Since the VIX is a tradable asset, by the fundamental pricing theorem in derivative theory, its future value process under the $\mathbb{Q}$-measure is a martingale. Then, the conditional variance $Var_t^{\mathbb{Q}}(VIX_{t+T_1\to t+T_1+T_2})$ equals $\left(FVIX^2_{t,t+T_1\to t+T_1+T_2}\right)\sigma_t^2 T_1$, where σ_t is the implied volatility of the at-the-money VIX option. Therefore,

$$\mathbb{E}_t^{\mathbb{Q}}[VIX^2_{t+T_1\to t+T_1+T_2}] = FVIX^2_{t,t+T_1\to t+T_1+T_2}(1+\sigma_t^2 T_1). \tag{A7}$$

Plug back into Equation (A6) and we obtain

$$F_t^2(1+\sigma_t^2 T_1) \sim \frac{1}{T_2}\left(\mathbb{E}^{\mathbb{Q}}_{t}\left[\left(\frac{R}{R_f}\right)^2\right]-1\right), \tag{A8}$$

where $F_t = FVIX_{t,t+T_1\to t+T_1+T_2}$ denotes the futures prices on VIX index.

Step 3. We derive the result for $\mathbb{E}_t^{\mathbb{Q}}[R^n]$ for $n \geqslant 3$. By the n-th order approximation of $\log(1+x)$, we have

$$\log(1+x) \sim \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i} x^i.$$

For $x = \frac{R}{R_f} - 1$, we obtain

$$\log\left(\frac{R}{R_f}\right) \sim \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i} \left(\frac{R}{R_f} - 1\right)^i.$$

By using the same method in Step 2, we have

$$\mathbb{E}_{t+T_1}^{\mathbb{Q}}\left[\log(\frac{R}{R_f})\right] \sim \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i} \mathbb{E}_{t+T_1}^{\mathbb{Q}}\left[\left(\frac{R}{R_f} - 1\right)^i\right].$$

Since

$$VIX_{t+T_1\to t+T_1+T_2}^2 = -\frac{2}{T_2}\mathbb{E}_{t+T_1}^{\mathbb{Q}}\left[\log(\frac{R}{R_f})\right] \sim \frac{2}{T_2}\sum_{i=1}^{n} (-1)^{i} \frac{1}{i} \mathbb{E}_{t+T_1}^{\mathbb{Q}}\left[\left(\frac{R}{R_f} - 1\right)^i\right],$$

By taking expectation conditional on t, the iterated law of expectation implies

$$\mathbb{E}_t[VIX_{t+T_1\to t+T_1+T_2}^2] \sim \frac{2}{T_2}\sum_{i=1}^{n} (-1)^{i} \frac{1}{i} \mathbb{E}_{t}^{\mathbb{Q}}\left[\left(\frac{R}{R_f} - 1\right)^i\right],$$

and we obtain,

$$\frac{T_2}{2} F_t^2 \left(1+\sigma^2 T_1\right) \sim \sum_{i=1}^{n} (-1)^{i} \frac{1}{i} \mathbb{E}_{t}^{\mathbb{Q}}\left[\left(\frac{R}{R_f} - 1\right)^i\right], \quad n \geqslant 3. \tag{A9}$$

□

Approximation error: We next explain why this approximation is sufficiently tight for empirical applications. For simplicity, we use $x = \frac{R_{t,t\to t+T}}{R_{ft,t\to t+T}} - 1$. Let $a \equiv \sup_x |\log(1+x) - (x - \frac{x^2}{2})|$

for all possible scenarios of x. The number a is very small in magnitude because x is closes to zero. Moreover, for any $c > 0$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\left|\log(1+x)-(x-\frac{x^2}{2})\right|\right] &= \mathbb{E}^{\mathbb{Q}}\left[\left|\log(1+x)-(x-\frac{x^2}{2})\right| : |x| \leqslant c\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\left|\log(1+x)-(x-\frac{x^2}{2})\right| : |x| > c\right] \\
&\leqslant \frac{c^3}{3} + \mathbb{E}^{\mathbb{Q}}\left[\left|\log(1+x)-(x-\frac{x^2}{2})\right| : |x| > c\right] \\
&\leqslant \frac{c^3}{3} + aP(|x| > c).
\end{aligned}$$

Clearly, the smaller the parameter c, the smaller the first term $\frac{c^3}{3}$. Although the probability $P(|x| \geqslant c)$ can become larger given a smaller value of c, this probability itself is usually very small. In total, the upper bound of $\mathbb{E}^{\mathbb{Q}}\left[|\log(1+x)-(x-\frac{x^2}{2})|\right]$ is very small.

Numerically, if choose $|x| \leqslant 1\%$ for the monthly return (annual return bound is 12 percent), and the average VIX is 15%, then

$$\mathbb{E}_t^{\mathbb{Q}}\left[\left|\log(1+x)-(x-\frac{x^2}{2})\right|\right] \leqslant \frac{1}{3}(0.01)^3,$$

and

$$\left|\mathbb{E}_t^{\mathbb{Q}}[\log(1+x)]\right| = \frac{T}{2}VIX^2 = \frac{1}{2\times 12}(0.15)^2.$$

Therefore,

$$\left|\frac{\mathbb{E}_t^{\mathbb{Q}}\left[\left|\log(1+x)-(x-\frac{x^2}{2})\right|\right]}{\mathbb{E}_t^{Q}[\log(1+x)]}\right| \leqslant \frac{1}{3}(0.01)^3 \times (2\times 12)\frac{1}{0.15^2} = 0.04\%.$$

If we choose a large number for the month return, $|x| \leqslant 2\%$, which means the annual return is bounded between $[-24\%, 24\%]$, and VIX = 20%, then

$$\left| \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\left|\log(1+x) - (x - \frac{x^2}{2})\right|\right]}{\mathbb{E}_t^{\mathbb{Q}}[\log(1+x)]} \right| \leqslant 0.16\%.$$

Therefore, the approximation formula is sufficiently accurate for the market data.

Appendix B Proof of Proposition 4.1

Before proving Proposition 4.1, we prove two results first. The first one presents an alternative expression of forward return in terms of expected value of future options' values. The second one is on a relationship between option gamma and strike gamma for a general option.

Proposition Appendix B.1. *Suppose that interest rates are deterministic. Then*

$$\mathbb{E}_t[R_{t+1\to t+2}] = \frac{2}{R_{f,t\to t+1} S_t} \int_0^{\infty} \mathbb{E}_t^{\mathbb{Q}} \left[\frac{C_{t+1}(S_{t+1}, K)}{S_{t+1}} \right] dK, \tag{B1}$$

where

- S_t = *underlying index price observed at time* t*;*
- C_{t+1} = *the call option price at time* $t+1$.

Proof. Suppose that interest rates are deterministic. By Equation (1), the expected future return under the real-world probability measure $\mathbb{P}$ can be written as

$$\mathbb{E}_t[R_{t+1\to t+2}] = \frac{1}{R_{f,t\to t+2}} \mathbb{E}_t^{\mathbb{Q}} \left[(R_{t+1\to t+2})^2 \times R_{t\to t+1} \right], \tag{B2}$$

$$= \frac{1}{R_{f,t\to t+2}} \mathbb{E}_t^{\mathbb{Q}} \left\{ \mathbb{E}_{t+1}^{Q} \left[(R_{t+1\to t+2})^2 \right] \times R_{t\to t+1} \right\}, \tag{B3}$$

where

$$\mathbb{E}^{\mathbb{Q}}_{t+1}\left[(R_{t+1\to t+2})^2\right]=\mathbb{E}^{\mathbb{Q}}_{t+1}\left[\left(\frac{S_{t+2}}{S_{t+1}}\right)^2\right]=\frac{1}{S^2_{t+1}}\mathbb{E}^{\mathbb{Q}}_{t+1}\left[S^2_{t+2}\right]. \tag{B4}$$

Plug back into the equation and we have

$$\mathbb{E}_t[R_{t+1\to t+2}]=\left(\frac{1}{R_{f,t\to t+2}}\right)\left(\frac{1}{S_t}\right)\mathbb{E}^{Q}_t\left\{\frac{1}{S_{t+1}}\mathbb{E}^{Q}_{t+1}\left[S^2_{t+2}\right]\right\}. \tag{B5}$$

We use Equation (3) for $\mathbb{E}^{Q}_{t+1}\left[S^2_{t+2}\right]$, obtaining

$$\begin{aligned}\frac{1}{R_{f,t+1\to t+2}}\mathbb{E}^{Q}_{t+1}\left[S^2_{t+2}\right]&=2\int_0^\infty\frac{1}{R_{f,t+1\to t+2}}\mathbb{E}^{Q}_{t+1}\left[(S_{t+2}-K)^+\right]dK &\text{(B6)}\\ &=2\int_0^\infty C_{t+1}(S_{t+1},K)dK, &\text{(B7)}\end{aligned}$$

where $C_{t+1}(S_{t+1},K)$ denotes the price of a call option at time $t+1$ that will expire at time $t+2$ with a strike price K.

Then, by Fubini's theorem, we have,

$$\begin{aligned}\mathbb{E}_t[R_{t+1\to t+2}]&=\left(\frac{1}{R_{f,t\to t+2}}\right)\left(\frac{1}{S_t}\right)\mathbb{E}^{Q}_t\left\{\frac{1}{S_{t+1}}\left[2R_{f,t+1\to t+2}\int_0^\infty C_{t+1}(S_{t+1},K)dK\right]\right\}\\ &=\frac{2}{R_{f,t\to t+1}S_t}\int_0^\infty\mathbb{E}^{Q}_t\left[\frac{C_{t+1}(S_{t+1},K)}{S_{t+1}}\right]dK. &\text{(B8)}\end{aligned}$$

□

The following result could be known as folklore in derivative literature. However, since we do not find an appropriate reference for this result, we present its complete proof.

Lemma Appendix B.1. *Let C'' denote the second-order partial derivative of call option price with respect to the underlying price, and $\ddot{C}$ the second-order partial derivative with respect to the strike,* *then we have*

$$S^2C''(S,K) = K^2\ddot{C}(S,K). \tag{B9}$$

Proof. Let C' denote the partial derivative of call option price with respect to (w.r.t.) the underlying price, $\dot{C}$ the partial derivative w.r.t. strike, and $\dot{C}'$ the second-order partial derivative w.r.t. strike and underlying.

We first demonstrate that $C(S,K)$ is homogeneous of degree 1. In other words, $C(aS,aK) = aC(S,K)$, for all real numbers $a > 0$. To see it, by the risk-neutral pricing equation,

$$C(S,K) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+ | S_t = S\right]. \tag{B10}$$

Using the formula $(ax)^+ = ax^+$, for all x and $a > 0$, the payoff is $(aS_T - aK)^+ = a(S_T - K)^+$. Then by the risk-neutral pricing equation again,

$$C(aS,aK) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(aS_T - aK)^+ | aS_t = aS\right] = ae^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+ | S_t = S\right] = aC(S,K). \tag{B11}$$

Accordingly, for any $a, b > 0$, we have

$$abC(S,K) = C(abS, abK). \tag{B12}$$

Take $\frac{\partial}{\partial a}$ on both sides of Eq. (B12) and set $a = 1$

$$bC(S,K) = bSC'(bS,bK) + bK\dot{C}(bS,bK). \tag{B13}$$

First, evaluate Equation (B13) at $b = 1$

$$C(S,K) = SC'(S,K) + K\dot{C}(S,K). \tag{B14}$$

Take partial derivative $\frac{\partial}{\partial K}$

$$\dot{C}(S,K) = S\dot{C}'(S,K) + K\ddot{C}(S,K) + \dot{C}(S,K), \tag{B15}$$

and we obtain

$$S\dot{C}'(S,K) + K\ddot{C}(S,K) = 0. \tag{B16}$$

Second, take first $\frac{\partial}{\partial a}$ on both sides of Equation (B13), and then $\frac{\partial}{\partial b}$ on the resulting equation, and set $a = b = 1$, we obtain

$$C(S,K) = SC'(S,K) + S^2C''(S,K) + 2S\dot{C}'(S,K)K + K\dot{C}(S,K) + K^2\ddot{C}(S,K). \tag{B17}$$

We next equate the right-hand sides of Equations (B14) and (B17) and obtain

$$S^2C''(S,K) + 2S\dot{C}'(S,K)K + K^2\ddot{C}(S,K) = 0. \tag{B18}$$

Plug the Equation (B16) into the equation above,

$$S^2C''(S,K) = K^2\ddot{C}(S,K), \tag{B19}$$

and we obtain Lemma Appendix B.1. □

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1

To compute the right-hand side of Equation (B8), and thus $\mathbb{E}_t[R_{t+1\to t+2}]$, write

$$C(S,K) = \frac{1}{R_f}\int_K^\infty (z-K)q(z|S)dz, \tag{B20}$$

where $q(\cdot)$ is the conditional density of S_{t+1} under the risk-neutral probability measure $\mathbb{Q}$ and R_f denotes the gross risk-free return. Now let $\dot{C}$ denote the partial derivative with respect to strike.

Then

$$\dot{C}(S,K) = \frac{1}{R_f}(\int_0^K q(z|S)dz - 1). \tag{B21}$$

Write $\Pi(K|S) = \int_0^K q(z|S)dz$ for the conditional distribution under $\mathbb{Q}$, that is, $\Pi(K|S) = Q(S_{t+1} \leqslant K|S_t = S)$. Then, $\Pi(K|S) = 1 + R_f\dot{C}(S,K)$, and $d\Pi(K|S) = R_f\ddot{C}(S,K)dK$.

Hence,

$$\begin{aligned}
\int_0^\infty \mathbb{E}_t^Q\left[\frac{C_{t+1}(S_{t+1},K)}{S_{t+1}}\right] dK &= \int_0^\infty \int_0^\infty \frac{C_{t+1}(L,K)}{L}\left(R_{f,t\to t+1}\ddot{C}_t(S_t,L)\,dL\right) dK \\
&= R_{f,t\to t+1}\int_0^\infty \int_0^\infty \frac{C_{t+1}(L,K)}{L}\ddot{C}_t(S_t,L)\,dL\,dK \\
&= R_{f,t\to t+1}S_t^2\int_0^\infty \int_0^\infty \frac{C_{t+1}(L,K)}{L}\frac{C_t^{''}(S_t,L)}{L^2}\,dL\,dK,
\end{aligned} \tag{B22}$$

where the last line substitutes gamma for strike-gamma using $C''(S,K) = \frac{K^2}{S^2}\ddot{C}(S,K)$, as specified by Lemma Appendix B.1.

Plug back into Equation (B8)

$$\mathbb{E}_t[R_{t+1\to t+2}] = 2S_t\int_0^\infty \frac{C_t^{''}(S_t,L)}{L^3}\left(\underbrace{\int_0^\infty C_{t+1}(L,K)\,dK}_{\text{inside-integral, } I(L)}\right) dL, \tag{B23}$$

and we obtain Proposition 4.1. □

Appendix C Nonzero Risk-neutral Relation

In this section, we provide a simple example to demonstrate that the risk-neutral correlation between the spot return and the future return square, $corr_t^{\mathbb{Q}}\left[R_{t\to t+1}, R^2_{t+1\to t+2}\right]$, can be nonzero.

At time $t = 0,1,2$, let the risk-free rate of return be zero, and the risky asset returns during the two consecutive periods be $R_{t\to t+1} = R_1 = 1+\varepsilon$ and $R_{t+1\to t+2} = R_2 = 1+\varepsilon\eta$, respectively. Suppose $\mathscr{F}_1$ is generated by ε, and $\mathscr{F}_2$ is by $\{\varepsilon,\eta\}$, where both ε and η are mean zero and independent of one another. Immediately, we have

$$\mathbb{E}[R_1] = 1 + \mathbb{E}[\varepsilon] = 1, \tag{C1}$$

$$\mathbb{E}_1[R_2] = 1 + \mathbb{E}_1[\varepsilon\eta] = 1 + \varepsilon\mathbb{E}_1[\eta] = 1, \tag{C2}$$

since $\mathbb{E}_1[\eta] = \mathbb{E}[\eta] = 0$. Therefore, the conditional expectation operator, $\mathbb{E}[.]$, is under the risk-neutral probability measure $\mathbb{Q}$.

We next compute the (risk-neutral) covariance between the spot return and the future return square, $Cov(R_1, R_2^2)$. First,

$$\mathbb{E}\left[R_2^2\right] = \mathbb{E}\left[1 + 2\varepsilon\eta + \varepsilon^2\eta^2\right] = 1 + \mathbb{E}\left[\varepsilon^2\eta^2\right] \tag{C3}$$

Second,

$$\mathbb{E}\left[R_1R_2^2\right] = \mathbb{E}\left[(1+\varepsilon)(1+2\varepsilon\eta+\varepsilon^2\eta^2)\right] = \mathbb{E}\left[1 + 2\varepsilon\eta + \varepsilon^2\eta^2\varepsilon + 2\varepsilon^2\eta + \varepsilon^3\eta^2\right], \tag{C4}$$

$$= 1 + \mathbb{E}\left[\varepsilon^2\eta^2\right] + \mathbb{E}\left[\varepsilon^3\eta^2\right]. \tag{C5}$$

Hence,

$$Cov(R_1, R_2^2) = \mathbb{E}\left[R_1R_2^2\right] - \mathbb{E}[R_1]\,\mathbb{E}\left[R_2^2\right], \tag{C6}$$

$$= \mathbb{E}\left[\varepsilon^3\eta^2\right], \tag{C7}$$

$$= \mathbb{E}\left[\varepsilon^3\right]\mathbb{E}\left[\eta^2\right]. \tag{C8}$$

In other words,

$$corr(R_1, R_2^2) \neq 0 \quad \text{if and only if} \quad \mathbb{E}\left[\varepsilon^3\right] \neq 0. \tag{C9}$$

If we choose $\varepsilon \sim \eta$ (the same distribution), in theory, the risk-neutral correlation $corr(R_1, R_2^2)$ can be any number (with the same sign as $\mathbb{E}\left[\varepsilon^3\right]$), as long as $\mathbb{E}\left[\varepsilon^3\right] \neq 0$.

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Table 1: Summary statistics of VIX and VIX-derivatives

This table provides the summary statistics for VIX, VIX futures, and implied volatility of VIX options. The sample period is from March 26, 2004 (February 24, 2006) to December 31, 2019 for VIX futures (options).

		Mean	Std dev	p25	Median	p75	Skew	Kurt
Panel A: VIX index								
		18.869	9.013	13.150	16.210	21.490	2.556	11.963
Panel B: VIX futures prices								
Maturity (in months)								
1		19.417	8.157	14.300	16.883	22.100	2.308	9.896
2		20.144	7.328	15.350	17.800	23.000	1.946	7.809
3		20.767	6.596	16.286	18.558	23.702	1.610	6.072
4		21.072	6.260	16.682	18.925	24.107	1.428	5.097
6		21.578	5.835	17.355	19.504	24.669	1.179	4.024
9		22.039	5.874	17.993	20.130	25.579	0.518	3.880
Panel C: Implied volatility of VIX options								
Maturity (in months)								
1	Put	0.891	0.168	0.789	0.871	0.974	1.268	8.293
	Call	0.893	0.164	0.788	0.875	0.978	1.139	7.033
2	Put	0.790	0.111	0.716	0.795	0.858	0.453	6.279
	Call	0.789	0.110	0.714	0.794	0.857	0.444	5.125
3	Put	0.717	0.089	0.660	0.724	0.776	0.159	4.776
	Call	0.716	0.089	0.657	0.723	0.774	0.245	5.208
4	Put	0.668	0.078	0.616	0.673	0.720	0.070	3.692
	Call	0.667	0.078	0.613	0.673	0.719	0.269	5.711
6	Put	0.630	0.071	0.579	0.635	0.678	0.067	3.396
	Call	0.628	0.072	0.576	0.634	0.676	0.153	4.176
9	Put	0.617	0.073	0.568	0.622	0.667	0.003	3.247
	Call	0.615	0.074	0.565	0.622	0.665	-0.019	3.161

Table 2: Expected future one-month return from the VIX-derivatives

This table provide the summary statistics for the expected future one-month return from the VIX-derivatives. The maturities are 1, 2, 3, 4, 6 and 9 months. We report mean, median, standard deviation, skewness and kurtosis. Panel A, B, and C consider three different sample periods: (i) full sample: February 24, 2006−December 31, 2019; (ii) Bad times (NBER recessions): January 1, 2008−June 30, 2009; and (iii) Good times (post-NBER recessions): July 1, 2009−December 31, 2019. All results are annualized and expressed in percentage.

	Avg. Ret (%)	Std dev (%)	p25	p50	p75	Skew	Kurt
Panel A: Sample period: February 24, 2006−December 31, 2019							
Maturity (in months)							
1	5.891	5.202	2.813	4.557	6.887	3.810	22.653
2	6.254	4.419	3.386	5.160	7.392	3.077	16.245
3	6.602	3.802	3.993	5.680	7.845	2.530	12.536
4	6.853	3.553	4.350	5.951	8.102	2.155	9.718
6	6.818	3.371	4.386	5.780	8.204	1.895	7.380
9	7.197	3.891	5.093	6.412	9.116	0.837	4.739
Panel B: Bad times (NBER recessions): January 1, 2008−June 30, 2009							
Maturity (in months)							
1	14.952	10.142	7.527	9.864	20.475	1.359	4.103
2	14.071	7.860	7.959	10.163	19.375	1.131	3.365
3	13.178	6.205	8.286	10.262	18.021	1.075	3.290
4	12.915	5.440	8.528	10.385	17.486	0.868	2.660
6	12.329	4.801	8.435	9.718	17.007	0.712	2.086
9	11.859	4.436	8.549	9.702	16.067	0.997	4.787
Panel C: Good times (post-NBER recessions): July 1, 2009−December 31, 2019							
Maturity (in months)							
1	4.332	2.477	2.569	3.683	5.233	2.056	8.809
2	4.891	2.399	3.066	4.357	5.753	1.574	5.815
3	5.442	2.356	3.582	4.968	6.331	1.330	4.548
4	5.781	2.360	3.939	5.294	6.714	1.289	4.395
6	6.409	2.451	4.499	5.724	7.366	1.174	3.754
9	7.446	2.931	5.360	6.439	9.011	1.023	3.683

Table 3: Market autocorrelation on S&P 500 index from the derivatives

Panels A and C report the statistics of the conditional market autocorrelation,

$$corr_t(R_{t\to t+T_1}, R_{t+T_1\to t+T_1+T_2}), \tag{C10}$$

and the conditional correlation between two spot returns,

$$corr_t(R_{t\to t+T_1}, R_{t\to t+T_1+T_2}), \tag{C11}$$

on S&P 500 index from the derivatives, respectively. For the two correlation coefficients, we also compute them from historical stock returns in Panels B and D, respectively. The sample period is from February 24, 2006 to December 31, 2019. *** indicates significance at the 1% level.

T_1	1 month	2 months	3 months	4 months	6 months	9 months
T_2	1 month	1 month	1 month	1 month	1 month	1 month
Panel A: $corr_t(R_{t\to t+T_1}, R_{t+T_1\to t+T_1+T_2})$ from derivatives						
Mean	-0.209***	-0.279***	-0.362***	-0.313***	-0.268***	-0.257***
p25	-0.290	-0.364	-0.462	-0.383	-0.318	-0.346
p50	-0.196	-0.238	-0.346	-0.270	-0.251	-0.255
p75	-0.087	-0.151	-0.229	-0.194	-0.199	-0.204
Skew	-0.944	-0.645	0.252	-1.364	0.018	1.048
Kurt	5.467	7.696	8.711	5.814	8.116	8.160
Panel B: $corr_t(R_{t\to t+T_1}, R_{t+T_1\to t+T_1+T_2})$ by realized historical returns						
$\hat{\rho}$	0.093	0.044	0.079	0.118	0.035	0.029
Panel C: $corr_t(R_{t\to t+T_1}, R_{t\to t+T_1+T_2})$ from derivatives						
Mean	0.829***	0.944***	0.978***	0.982***	0.976***	0.971***
p25	0.734	0.912	1.000	1.000	1.000	1.000
p50	0.850	0.997	1.000	1.000	1.000	1.000
p75	0.963	1.000	1.000	1.000	1.000	1.000
Skew	-0.765	-8.093	-9.046	-4.847	-4.448	-3.833
Kurt	3.151	114.762	111.636	31.996	24.396	18.117
Panel D: $corr_t(R_{t\to t+T_1}, R_{t\to t+T_1+T_2})$ by realized historical returns						
$\hat{\rho}$	0.746***	0.841***	0.889***	0.919***	0.944***	0.961***

Table 4: Market timing

This table reports the investment value of timing the previous cumulative market excess return and conditional market autocorrelation. We consider six reversal signals, $\tilde{S}_{t,K}$, for K = 1, 2, 3, 4, 6, and 9 months such that,

$$\tilde{S}_{t,K}\left[r_{t-K\to t}, corr_{t-k}(r_{t-K\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-K\to t} > 0 \quad \& \quad corr_{t-k}(r_{t-K\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-K\to t} < 0 \quad \& \quad corr_{t-k}(r_{t-K\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $r_{t-K\to t}$ denote the cumulative excess returns over the past K months, and $corr_{t-k}(r_{t-K\to t}, r_{t\to t+1})$ is the conditional autocorrelation from the derivatives computed at time $t-K$.

The single timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$ takes a long position in the market when the one-month reversal signal, $\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right]$ equals one, and invests in the risk-free asset otherwise. The combination timing strategy, $\eta\left[\tilde{S}_{t,K}, \forall K; \xi\right]$ utilizes all six reversal signals, and takes a long position in the market only if at least ξ out of six reversal signals take values of ones. We consider ξ to be 1, 2, 3, 4, and 5.

Panel A and B consider two different out-of-sample periods: 1) full sample period; 2) NBER recession period: January 2008−June 2009. The average value, standard deviation, and return-loss are expressed in percentage, and the performance fee is in basis points. All results are annualized.

	Avg ex-Ret (%)	Std dev (%)	SRatio	CEQ	SRatio Diff	CEQ Diff	Ret-Loss (%)	Fee (bps)
Panel A: Full sample period								
Buy and hold	5.489	14.789	0.371	0.044				
$\eta\left[\tilde{S}_{t,1}\right]$	4.433	9.616	0.461	0.040	0.090	-0.004	-0.863	-74.433
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 2\right]$	2.389	11.175	0.214	0.018	-0.157	-0.026	1.759	-286.856
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 3\right]$	1.997	9.986	0.200	0.015	-0.171	-0.029	1.709	-319.872
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 4\right]$	1.733	9.343	0.185	0.013	-0.186	-0.031	1.735	-343.248
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 5\right]$	3.347	7.166	0.467	0.031	0.096	-0.013	-0.687	-172.763
Panel B: NBER recessions: January 2008−June 2009								
Buy and hold	-32.304	25.565	-1.264	-0.356				
$\eta\left[\tilde{S}_{t,1}\right]$	-7.027	14.420	-0.487	-0.081	0.776	0.275	-11.194	2630.850
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 2\right]$	-23.918	20.746	-1.153	-0.261	0.111	0.095	-2.297	890.633
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 3\right]$	-11.842	18.780	-0.631	-0.136	0.633	0.220	-11.888	2116.325
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 4\right]$	-11.842	18.780	-0.631	-0.136	0.633	0.220	-11.888	2116.325
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 5\right]$	-0.456	11.728	-0.039	-0.011	1.225	0.344	-14.364	3304.994

Table 5: Market timing by VIX-Gamma

This table reports the single timing strategy based on the 1-month reversal signal identified from the VIX-Gamma approach. Panel A and B consider two different out-of-sample periods: 1) full sample period; 2) NBER recession period: January 2008−June 2009. The average value, standard deviation, and return-loss are expressed in percentage, and the performance fee is in basis points. All results are annualized. The statistical significance of the Sharpe ratio difference (SRatio Diff) and certainty equivalent return difference (CEQ Diff) are evaluated based on *p*-values using the Jobson and Korkie's (1981) methodology described in Section 2 of DeMiguel, Garlappi, and Uppal (2009). ** and * indicate significance at the 5% and 10% levels, respectively.

	Avg ex-Ret (%)	Std dev (%)	SRatio	CEQ	SRatio Diff	CEQ Diff	Ret-Loss (%)	Fee (bps)
Panel A: Full sample period								
Buy and hold	5.489	14.789	0.371	0.044				
VIX-Gamma	5.659	11.322	0.500	0.050	0.129*	0.006**	-1.456	39.453
Panel B: NBER recessions: January 2008−June 2009								
Buy and hold	-32.304	25.565	-1.264	-0.356				
VIX-Gamma	6.538	19.480	0.336	0.046	1.599	0.402	-31.153	3948.062

Table 6: Implied risk-neutral correlation

This table reports the average value of the risk-neural correlation, $\theta_t = corr_t^{\mathbb{Q}}\left(R_{t\to t+1}, R^2_{t+1\to t+2}\right)$, that are "calibrated" from the VIX-Gamma approach. We compute the average values either by moving-averages (MAs) or by the calendar years.

By MAs:	6-month	1-year	3-year	5-year	Overall
	-0.046	-0.029	0.006	0.013	-0.060
By years:	2006−2009	2010−2011	2012−2014	2015−2017	2018−2019
	-0.059	-0.281	0.049	-0.092	0.040

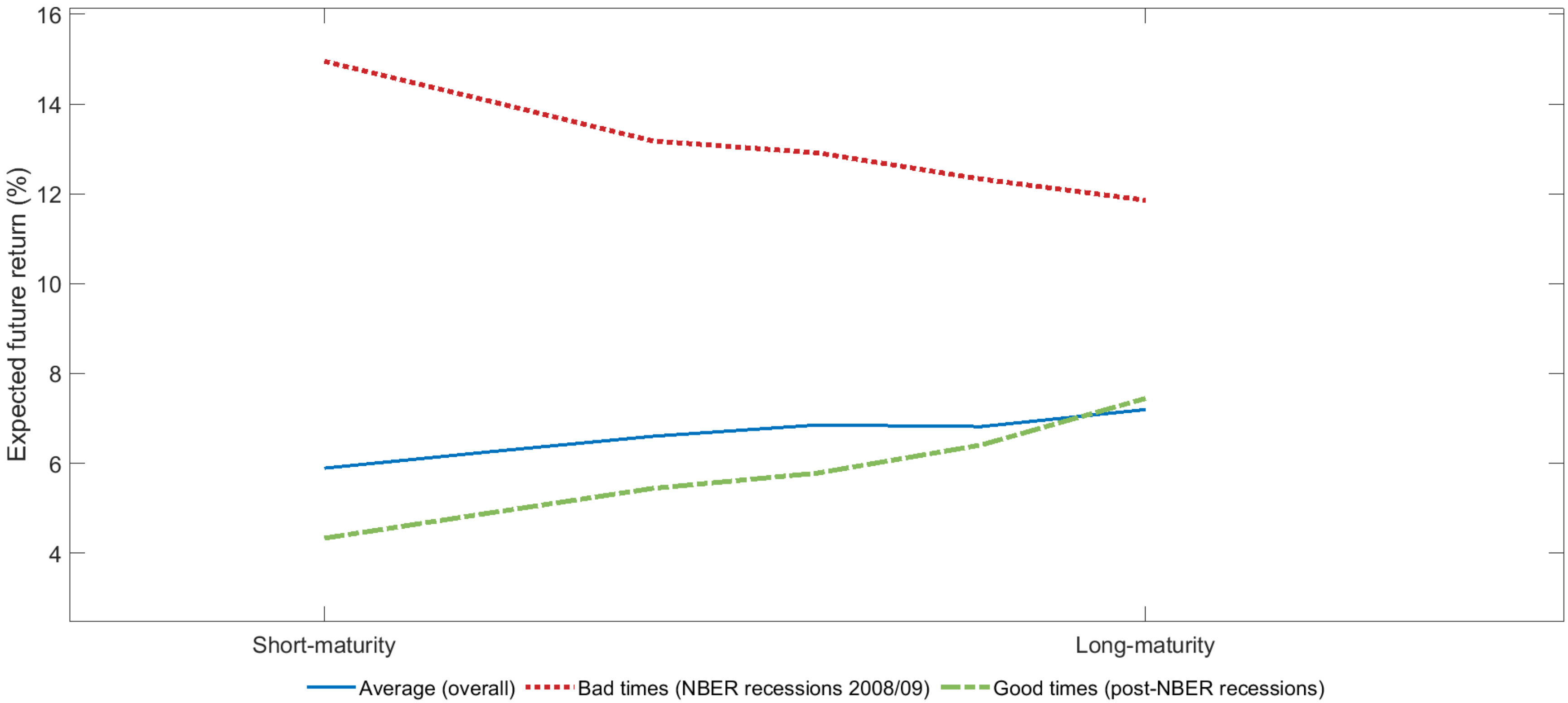


Figure 1: The term structure of expected future one-month return

This figure plots the term structure of the expected future one-month returns by VIX-derivatives. The figure shows the unconditional average return (solid line), the average return in bad times from January 2008 to June 2009 during the NBER recessions (dashed line), and the average return in good times during the post NBER recessions (dash-dotted line).

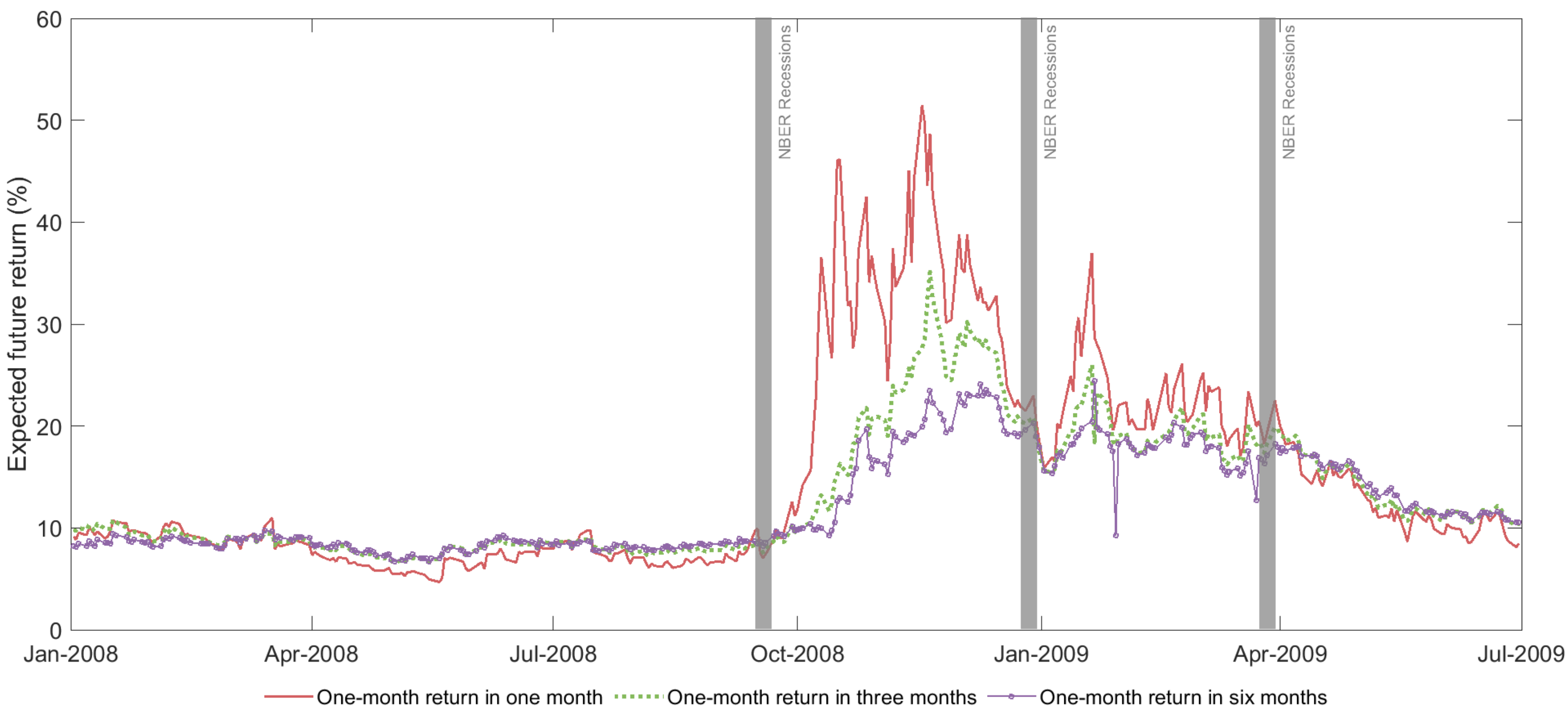


Figure 2: Expected future one-month returns during the NBER recessions

This figure plots the expected one-month returns in one, three, and six months by VIX-derivatives during the NBER recessions from January 1, 2008 to June 30, 2009. All results are annualized and expressed in percentage.

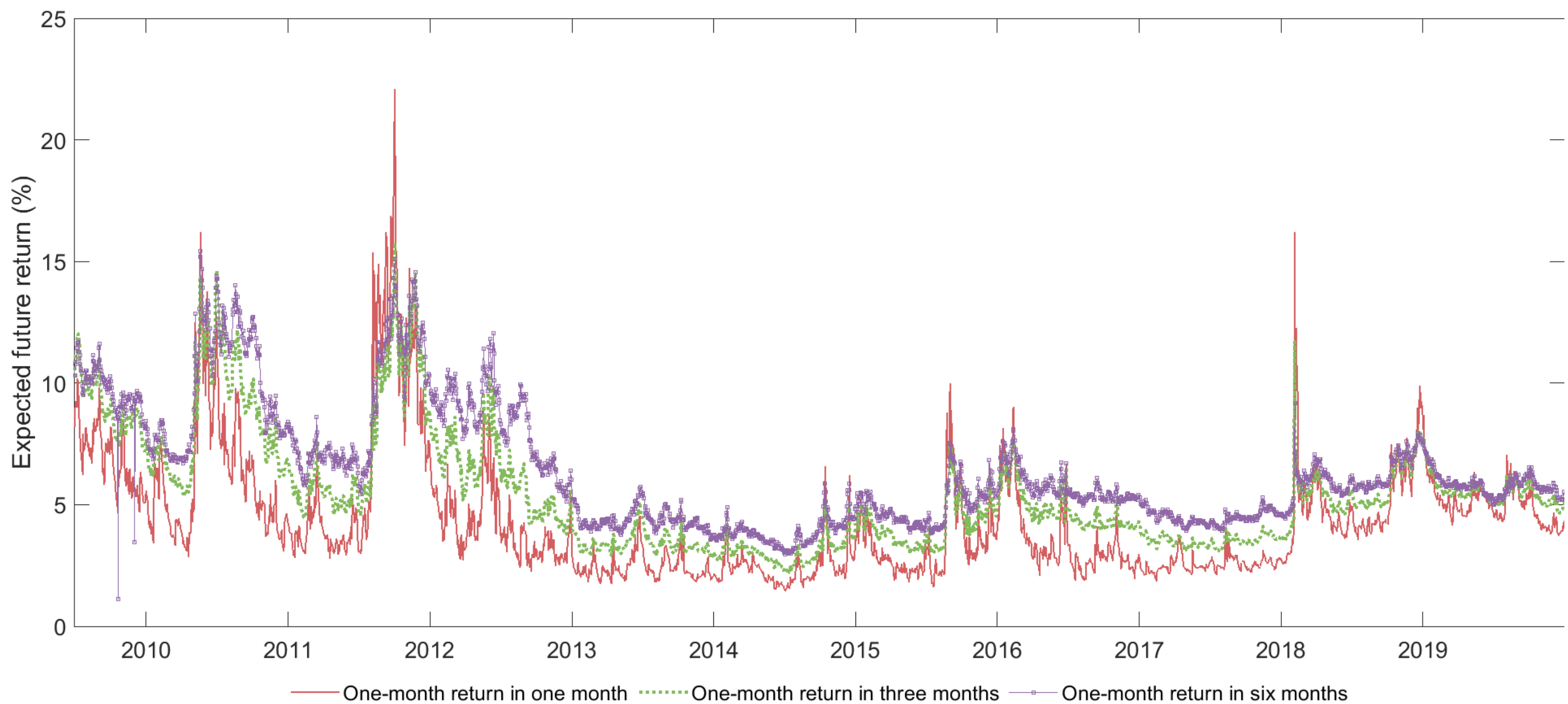


Figure 3: Expected future one-month returns post the NBER recessions

This figure plots the expected one-month returns in one, three, and six months by VIX-derivatives during the *post* NBER recession period from July 1, 2009 to December 31, 2019. All results are annualized and expressed in percentage.

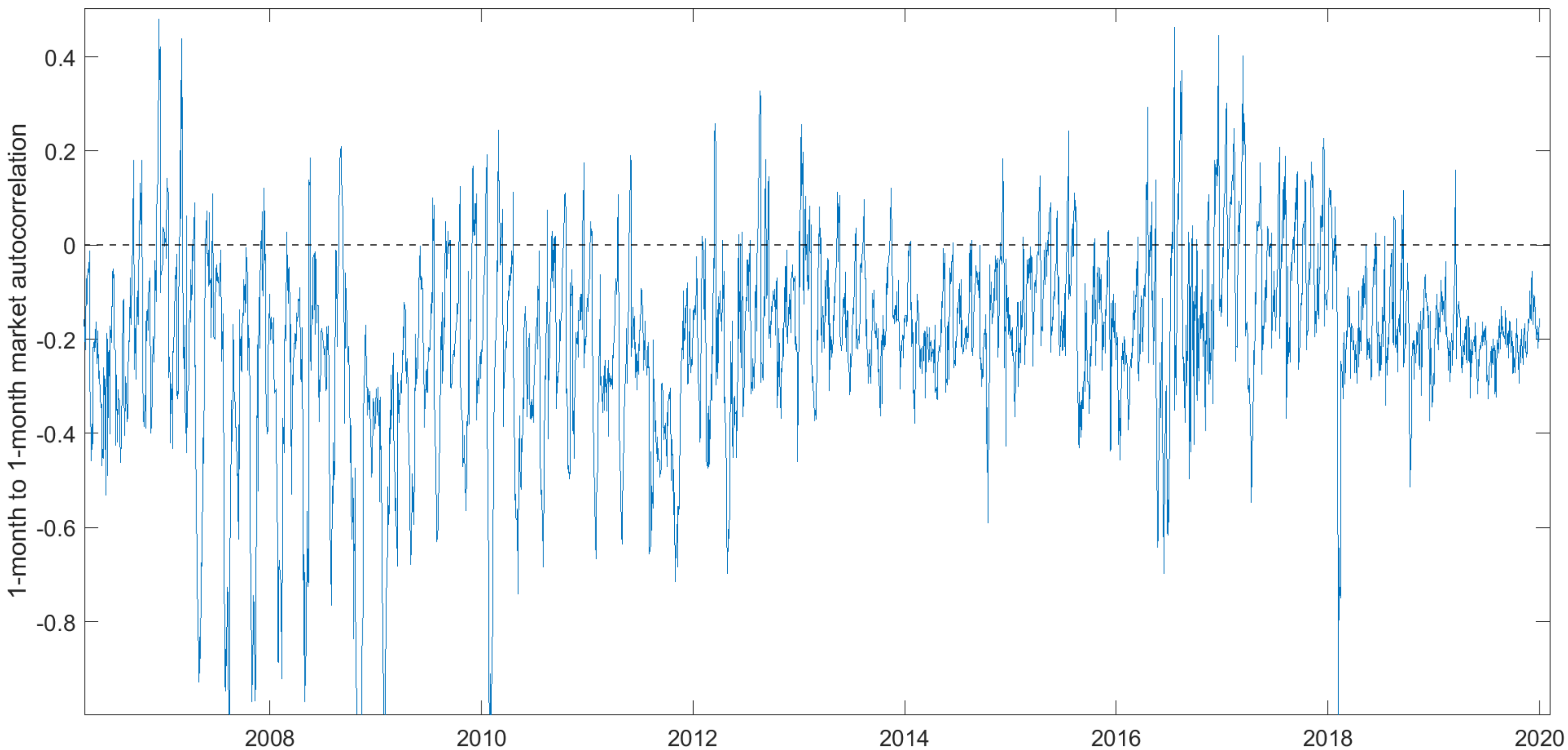
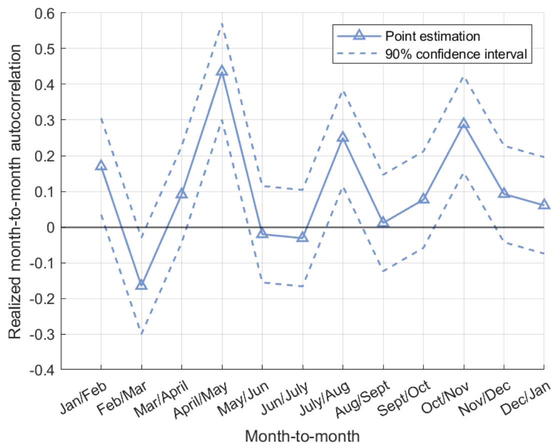
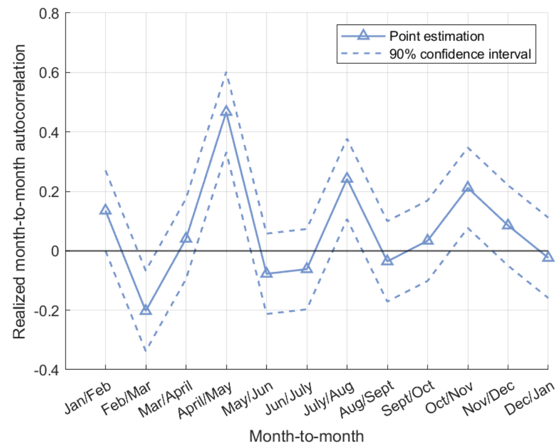


Figure 4: Market autocorrelation on S&P 500 index from VIX-derivatives

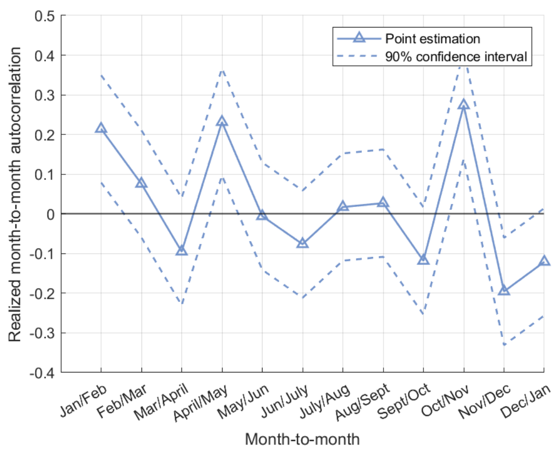
This figure plots the real-time forward-looking 1-month to 1-month market autocorrelation, $corr_t(R_{t\rightarrow t+1mo}, R_{t+1mo\rightarrow t+2mo})$, on S&P 500 index recovered from VIX-derivatives.



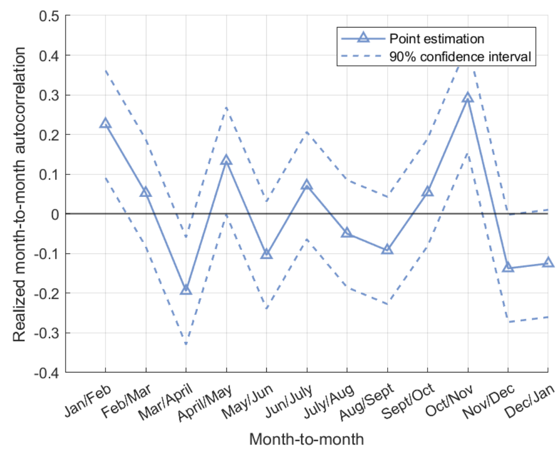
(a) Sample period 1871 - 2019



(b) Sample period 1920 - 2019



(c) Sample period 1970 - 2019



(d) Sample period 1990 - 2019

Figure 5: Realized market autocorrelation between adjacent calendar months

This figure plots the realized month-to-month autocorrelation of the S&P 500 monthly returns between two consecutive months. The area between the dotted line represents the 90% confidence interval for the sample autocorrelation by assuming the standard error equals one over the square root of the sample size. We consider four time periods: (a) 1871 – 2019, (b) 1920 – 2019, (c) 1970 – 2019, (d) 1990 – 2019. The data prior to January 1927 are obtained from Robert Shiller's website.

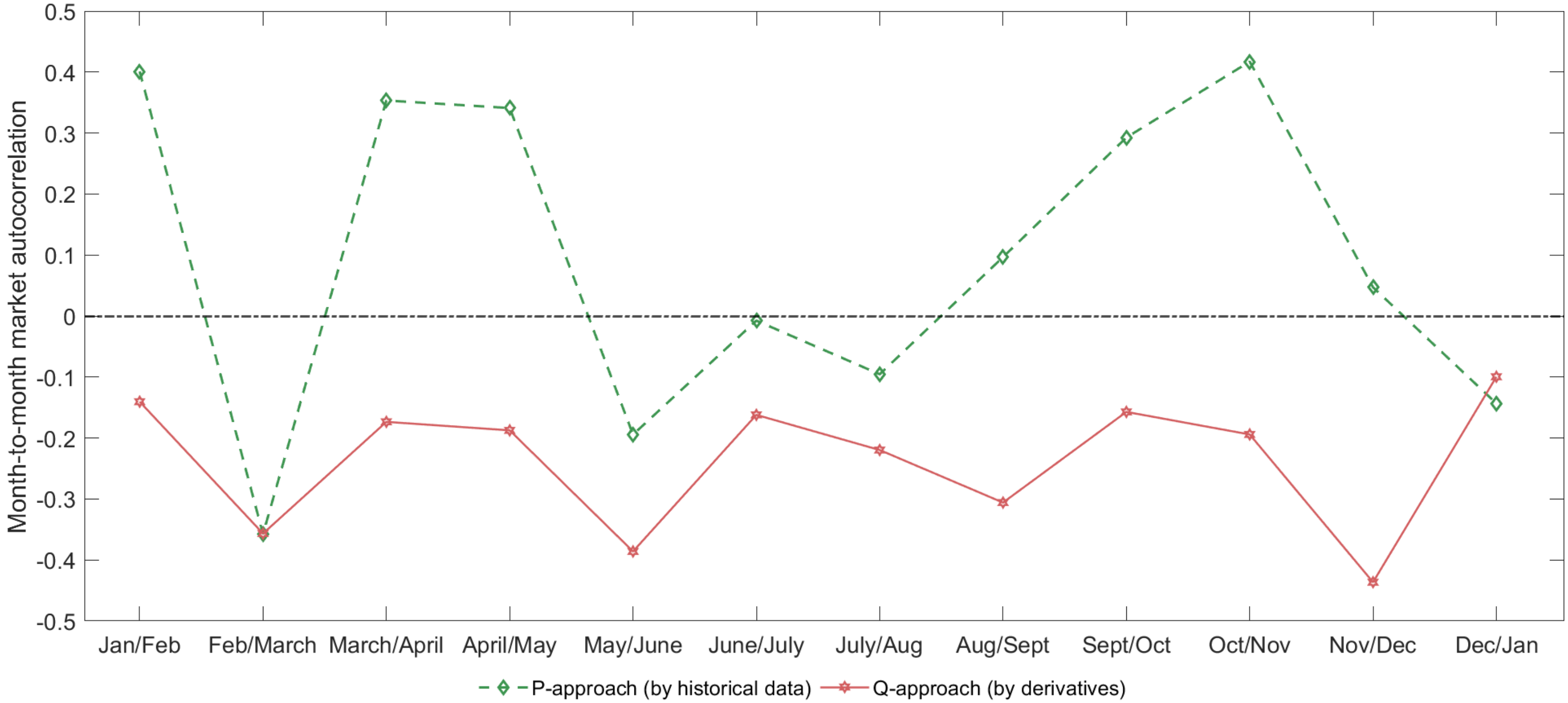


Figure 6: Month-to-month market autocorrelation by derivatives and historical data

This figure plots the month-to-month autocorrelation on S&P 500 index between two consecutive months. By historical return, we compute the sample autocorrelation using historical monthly return data; by VIX-approach, we compute $corr_t(R_{t \to t+1mo}, R_{t+1mo \to t+2mo})$ by derivative data on the first day of each month, and then take the average within January, February, ..., and December. The sample period is from 2006 to 2019.

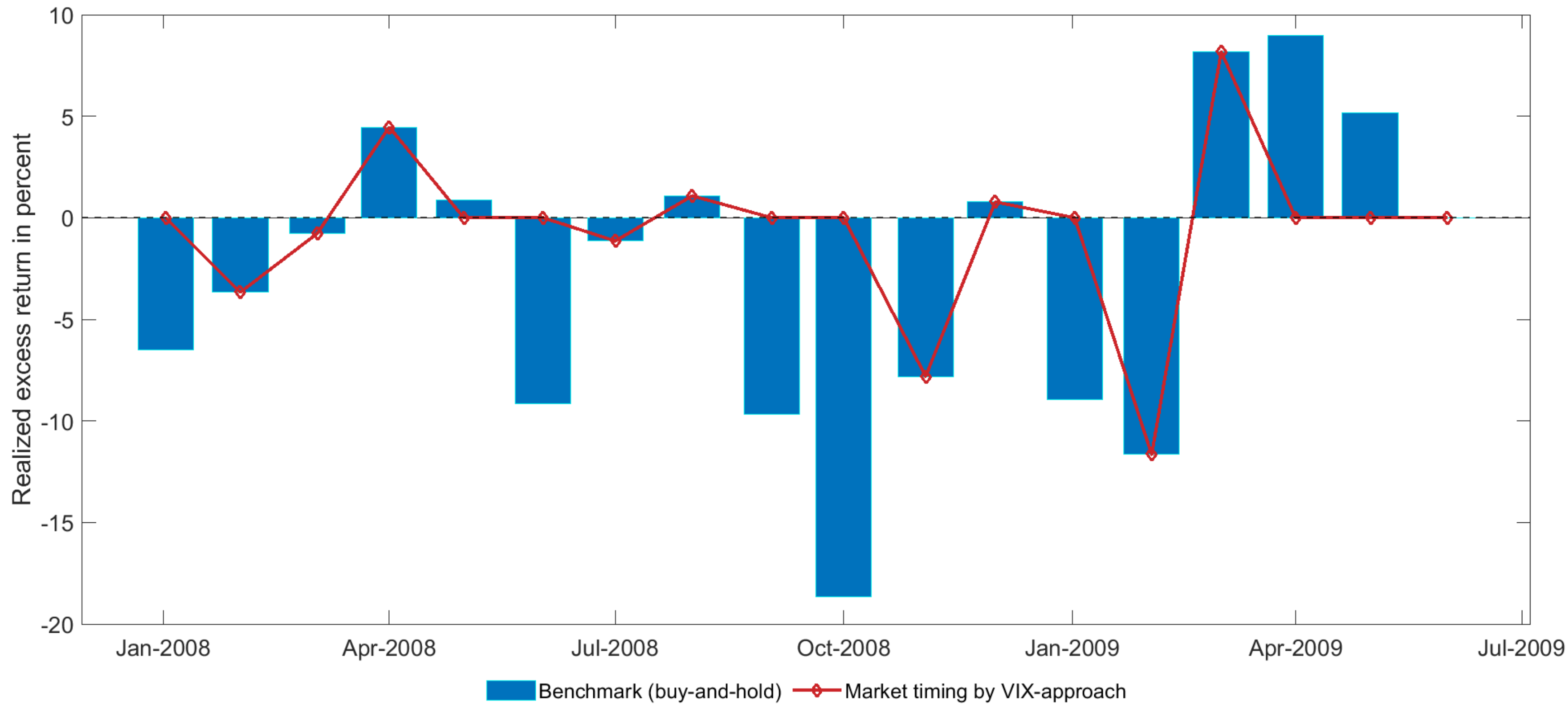
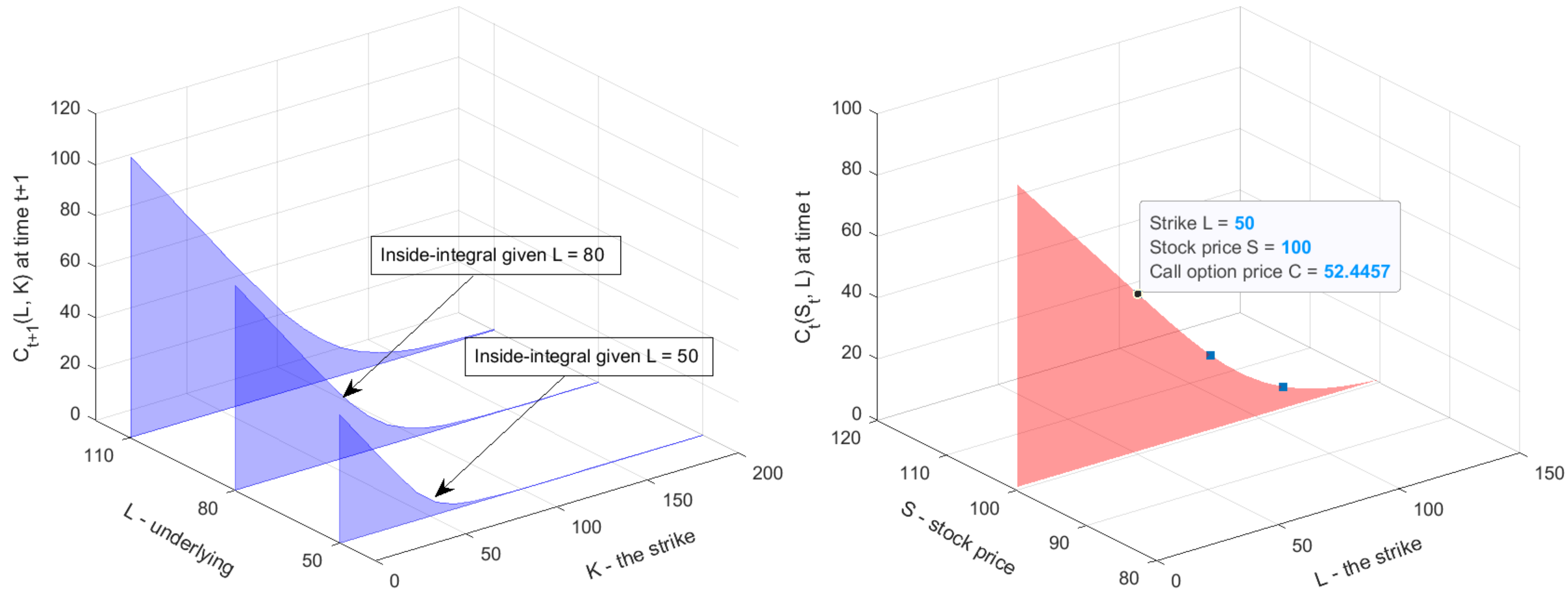


Figure 7: Market timing during NBER recessions

This figure plots the realized out-of-sample excess returns generated from either buy-and-hold strategy (benchmark) or the market timing strategy over the NBER recessions from January 2008 to June 2009. The market timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$ takes a long position in the market when the one-month reversal signal, $\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right]$ equals one, and invests in the risk-free asset otherwise. Mathematically,

$$\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-1\to t} > 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-1\to t} < 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $r_{t-1\to t}$ denote the cumulative excess returns over the past 1 month, and $corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})$ is the conditional autocorrelation from VIX-approach computed at time $t-1$.



(a) Call option at time $t+1$ **(b)** Call option at time t

Figure 8: Calculating the two-integral from call option prices

This figure illustrates the calculation of the two-integral in Proposition 4.1 where

$$\mathbb{E}_t[R_{t+1\to t+2}] = 2S_t \int_0^\infty \frac{C_t''(S_t,L)}{L^3} \left(\underbrace{\int_0^\infty C_{t+1}(L,K)\,dK}_{\text{inside-integral, } I_{t+1}(L)\text{, known at } t+1} \right) dL.$$

At time t in the right-side panel, we plot the call option price, $C_t(S_t,L)$ for a sequence of strike prices, $L \geqslant 0$, assuming $S_t = 100$, $r_f = 5\%$, $\sigma = 25\%$, and $T = 1$ year. At time $t+1$ in the left-side panel, we plot the call option prices, $C_{t+1}(L,K)$, given each L "observed" at time t in the right-side panel as the new underlying prices, and for a sequence of strikes, $K \geqslant 0$.

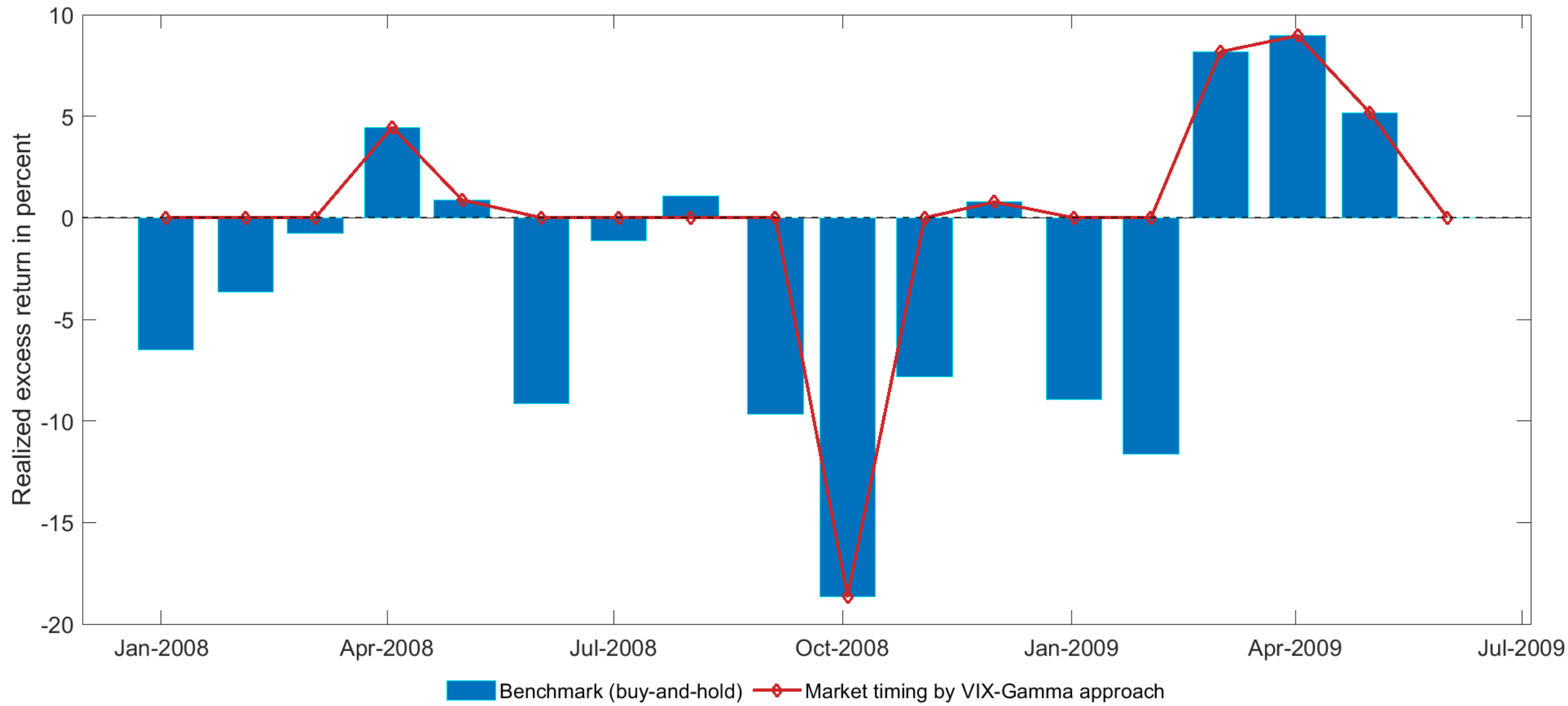


Figure 9: Market timing by VIX-Gamma approach during NBER recessions

This figure plots the realized out-of-sample excess returns generated from either buy-and-hold strategy (benchmark) or the market timing strategy over the NBER recessions from January 2008 to June 2009. The market timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$ takes a long position in the market when the one-month reversal signal, $\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right]$ equals one, and invests in the risk-free asset otherwise. Mathematically,

$$\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-1\to t} > 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-1\to t} < 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $r_{t-1\to t}$ denote the cumulative excess returns over the past 1 month, and $corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})$ is the conditional autocorrelation from VIX-Gamma approach computed at time $t-1$.

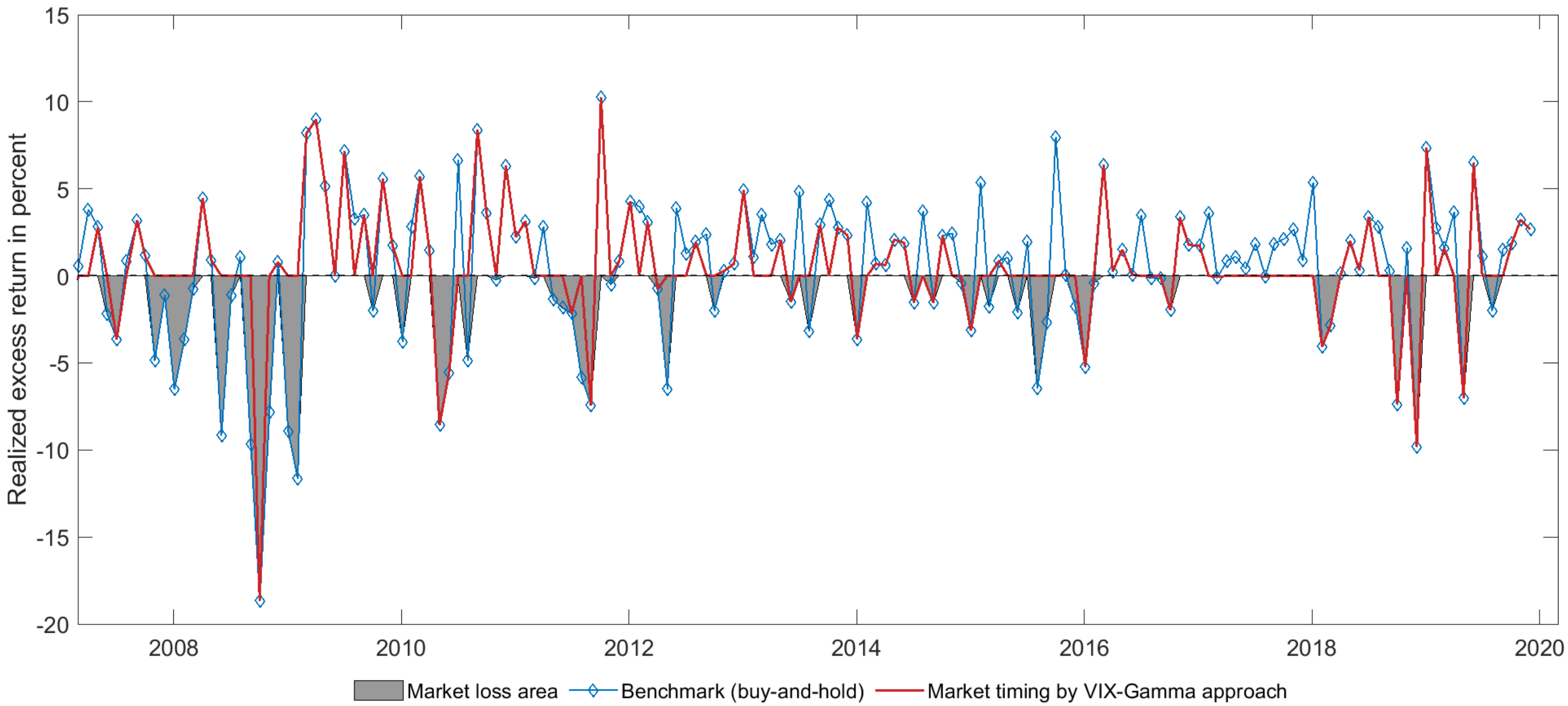


Figure 10: Market timing by VIX-Gamma approach

This figure plots the realized out-of-sample excess returns generated from either buy and hold strategy (benchmark) or market timing strategy over the out-of-sample evaluation period from 2006 to 2019. The market timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$ takes a long position in the market when the one-month reversal signal, $\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right]$ equals one, and invests in the risk-free asset otherwise. Mathematically,

$$\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-1\to t} > 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-1\to t} < 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $r_{t-1\to t}$ denote the cumulative excess returns over the past 1 month, and $corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})$ is the conditional autocorrelation from VIX-Gamma approach computed at time $t-1$.